A Generalized Fixed Point Theorem on Metric Spaces

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ABSTRACT
The aim of this paper is to obtain a related fixed point theorem on five metric spaces. This theorem generalizes and extends the results obtained in [3] from four metric spaces to five metric spaces. Further a generalization of the theorem 2.1 proved in [3] is obtained as corollary of our theorem.

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INTRODUCTION

The application of fixed point theorems are very important in diverge disciplines of mathematics, statistics, engineering and economics in dealing with problems related to approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, theory of integral equations, etc.

Main results
The main theorem is as follows:

Theorem 2.1: Let (X, d₁), (Y, d₂), (Z, d₃), (U, d₄) and (V, d₅) be complete metric spaces. Let T : X → Y, S : Y → Z, R : Z → U, Q : U → V and P : V → X be five mappings satisfying the following inequalities:

\[
d₁(PQRSy, PQRSTx) ≤ c \frac{f₁(x, y)}{g₁(x, y)} \quad (1)
\]

\[
d₂(TPQRSy, TPQRSy) ≤ c \frac{f₂(y, z)}{g₂(y, z)} \quad (2)
\]

\[
d₃(STPQu, STPQRz) ≤ c \frac{f₃(z, u)}{g₃(z, u)} \quad (3)
\]

\[
d₄(RSTPv, RSTQu) ≤ c \frac{f₄(u, v)}{g₄(u, v)} \quad (4)
\]

\[
d₅(QRSTx, QRSTx) ≤ c \frac{f₅(v, x)}{g₅(v, x)} \quad (5)
\]

for all \(x \in X, y \in Y, z \in Z, u \in U \) and \(v \in V \) for which \(g₁(x, y) ≠ 0, g₂(y, z) ≠ 0, g₃(z, u) ≠ 0, g₄(u, v) ≠ 0, g₅(v, x) ≠ 0 \), where \(0 ≤ c < 1 \) and

\[
f₁(x, y) = \max \{d₁(x, PQRSy)d₁(QRSTx), d₁(x, PQRSy)d₁(RSx, RSTx); d₁(x, PQRSTx)d₁(Sy, STx); d₁(x, PQRSTx)d₁(y, TPQRSy); d₁(x, PQRSy)d₁(y, Tx)\}.
\]

\[
f₂(y, z) = \max \{d₂(y, TPQRSy)d₁(PQRz, PQRSy); d₂(y, TPQRSy)d₁(Rz, QRSy); d₂(y, TPQRSy)d₁(z, STPQRz); d₂(y, TPQRSy)d₁(z, S₃)\}.
\]

\[
f₃(z, u) = \max \{d₃(z, STPQRz)d₁(TPQu, TPQRz); d₃(z, STPQRz)d₁(PQu, PQRz); d₃(z, STPQRz)d₁(Qu, QRz); d₃(z, STPQRz)d₁(u, RSTPQu); d₃(z, STPQRz)d₁(u, R₃)\}.
\]

\[
f₄(u, v) = \max \{d₄(u, RSTPQu)d₃(STPv, TPQu); d₄(u, RSTPQu)d₃(TPv, TPQu); d₄(u, RSTPQu)d₃(Pv, PPQu); d₄(u, RSTPQu)d₃(v, QRSTPv); d₄(u, RSTPQu)d₃(v, Qu)\}.
\]

\[
f₅(v, x) = \max \{d₅(v, QRSTPv)d₄(RSTx, RSTPv); d₅(v, QRSTPv)d₄(STx, STPv); d₅(v, QRSTPv)d₄(Tx, TPv); d₅(v, QRSTPv)d₄(x, PQRSx); d₅(v, QRSTPv)d₄(x, PV)\}.
\]

\[
g₁(x, y) = \max \{d₁(x, PQRSy); d₁(x, PQRSy); d₁(Tx, TPQRSy)\}.
\]

\[
g₂(y, z) = \max \{d₂(y, TPQRSy); d₂(y, TPQRSy); d₂(Rz, QRSy)\}.
\]

\[
g₃(z, u) = \max \{d₃(z, STPQRz); d₃(z, STPQRz); d₃(Qu, QRz)\}.
\]

\[
g₄(u, v) = \max \{d₄(u, RSTPQu); d₄(u, RSTPQu); d₄(u, RSTPQu)\}.
\]

\[
g₅(v, x) = \max \{d₅(v, QRSTPv); d₅(v, QRSTPv); d₅(x, PQRSx)\}.
\]
\( g_3(z, u) = \max \{d_3(z, STPu); d_3(z, STPQz); d_4(Rz, RSTPQu)\} \)
\( g_3(u, v) = \max \{d_3(u, RSTPu); d_3(u, RSTPQu); d_4(Qu, QRSTPv)\} \)
\( g_4(v, x) = \max \{d_4(v, VRSTx); d_4(v, VRSTPv); d_4(Pv, PQRSTx)\} \)

Then PQRST has a unique fixed point \( \alpha \in X \), TPQRS has a unique fixed point \( \beta \in Y \), STPQR has a unique fixed point \( \gamma \in Z \), RSTPQ has a unique fixed point \( \delta \in U \) and QRSTP has a unique fixed point \( \xi \in V \). Further, \( T\alpha = \beta \), \( SB = \gamma \), \( R\gamma = \delta \), \( QS = \xi \) and \( P\xi = \alpha \).

**Proof:** Let \( x_0 \in X \) be an arbitrary point. First we define the five sequences \( x_n, y_n, u_n, v_n \) and \( z_n \) of \( X, Y, Z, U \) and \( V \) respectively as follows:

\[
\begin{align*}
x_n &= (PQRST)^n x_0; \quad y_n = (Tx_{n-1}) \cdot \quad z_n = Sy_{n-1} \cdot \quad u_n = Rx_{n-1} \quad \text{and} \quad v_n = Qu_{n-1} \quad \forall \ n \in N.
\end{align*}
\]

We assume that \( x_n \neq x_{n+1}, y_n \neq y_{n+1}, z_n \neq z_{n+1}, u_n \neq u_{n+1}, v_n \neq v_{n+1} \quad \forall \ n \in N \). For otherwise if \( x_n = x_{n+1} \) for some \( n \), \( y_n = y_{n+2} \), \( z_n = z_{n+2} \), \( u_n = u_{n+2} \), \( v_n = v_{n+2} \) and we could put \( x_n = \alpha \), \( y_{n+1} = \beta \), \( z_{n+1} = \gamma \), \( u_{n+1} = \delta \), \( v_{n+1} = \xi \).

If \( y_n = y_{n+1} \), then \( z_n = z_{n+1} \), \( u_n = u_{n+1} \) and \( v_n = v_{n+1} \) and the later equalities imply that \( x_n = x_{n+1} \). If \( z_n = z_{n+1} \) then \( u_n = u_{n+1} \) and \( v_n = v_{n+1} \) and the later equalities again imply that \( x_n = x_{n+1} \) and \( y_n = y_{n+1} \). In a similar way if \( u_n = u_{n+1} \) or \( v_n = v_{n+1} \) then again \( x_n = x_{n+1} \). Taking \( z = z_{n+1} \), \( y = y_n \) in (2) we get

\[
d_2(y_n, y_{n+1}) = d_2(TPQRz_n, TPQRSy_n) \leq c \cdot \frac{f_2(y_n, z_{n+1})}{g_2(z_n, z_{n-1})}
\]

\[
= c \max \{d_2(y_n, y_{n+1})d_1(x_{n-1}, x_n); d_2(y_n, y_{n+1})d_3(y_{n-1}, y_n); d_2(y_n, y_{n+1})d_4(u_{n-1}, u_n);
\]

\[
d_2(y_n, y_{n+1})d_2(z_{n-1}, z_n); d_2(y_n, y_{n+1})d_3(z_{n-1}, z_n)
\]

\[
= c \max \{d_2(y_n, y_{n+1}) [d_1(x_{n-1}, x_n); d_3(y_{n-1}, y_n); d_4(u_{n-1}, u_n); d_3(z_{n-1}, z_n)]
\]

\[
d_2(y_n, y_{n+1})
\]

Thus \( d_2(y_n, y_{n+1}) \leq c \max \{d_1(x_{n-1}, x_n); d_3(z_{n-1}, z_n); d_4(u_{n-1}, u_n); d_3(v_{n-1}, v_n)\} \quad (6) \)

Taking \( u = u_{n+1} \) and \( z = z_n \) in (3) we get

\[
d_3(z_n, z_{n+1}) = d_3(STPQz_n, STPQz_{n+1}) \leq c \cdot \frac{f_3(z_n, u_{n-1})}{g_3(z_n, u_{n-1})}
\]

\[
= c \max \{d_3(z_n, z_{n+1})d_2(y_{n+1}, y_n); d_3(z_n, z_{n+1})d_1(x_{n-1}, x_n); d_3(z_n, z_{n+1})d_3(y_{n-1}, y_n); d_3(z_n, z_{n+1})d_4(u_{n-1}, u_n); d_3(z_n, z_{n+1})d_2(u_{n-1}, u_n)
\]

\[
= c \max \{d_3(z_n, z_{n+1}) [d_2(y_{n+1}, y_n); d_1(x_{n-1}, x_n); d_3(y_{n-1}, y_n); d_4(u_{n-1}, u_n)]
\]

\[
d_3(z_n, z_{n+1})
\]

Using (6) we get

\[
d_3(z_n, z_{n+1}) \leq c \max \{d_1(x_{n+1}, x_n); d_3(z_{n+1}, z_n); d_4(u_{n+1}, u_n); d_3(v_{n+1}, v_n)\} \quad (7) \)

Taking \( v = v_{n+1} \) and \( u = u_n \) in (4) we get

\[
d_4(u_n, u_{n+1}) = d_4(RSTPV_{n+1}, RSTPQu_n) \leq c \cdot \frac{f_4(u_n, v_{n-1})}{g_4(u_n, v_{n-1})}
\]

\[
= c \max \{d_4(u_n, u_{n+1})d_3(z_{n+1}, z_n); d_4(u_n, u_{n+1})d_2(y_{n+1}, y_n); d_4(u_n, u_{n+1})d_1(x_{n+1}, x_n);
\]

\[
d_4(u_n, u_{n+1})d_3(y_{n+1}, y_n); d_4(u_n, u_{n+1})d_3(v_{n+1}, v_n)
\]

\[
= c \max \{d_4(u_n, u_{n+1}) [d_3(z_{n+1}, z_n); d_2(y_{n+1}, y_n); d_1(x_{n+1}, x_n); d_3(v_{n+1}, v_n)]
\]

\[
d_4(u_n, u_{n+1})
\]
Letting $n$ 

Since $0$ 

inequalities

Continuing this process, by induction on inequalities (6), (7), (8), (9) and (10) we get the following: 

Using (6), (7) we get 

Taking $v = v_n$ and $x = x_n$ in (5) we get

Using (6), (7) and (8) we get 

Using (6) and (7) we get 

Using (6), (7), (8) and (9) we get 

Using (6), (7) and (8) we get 

Using (6), (7), (8), (9) and (10) we get the following inequalities 


d_1(x_n, x_{n+1}) \leq c \max \{ d_1(x_n, x_{n+1}); d_2(y_n, y_{n+1}); d_2(u_n, u_{n+1}); d_2(v_n, v_{n+1}) \} \quad (10) 

Continuing this process, by induction on inequalities (6), (7), (8), (9) and (10) we get the following inequalities

Since $0 \leq c < 1$, the sequences $< x_n >$, $< y_n >$, $< z_n >$, $< u_n >$ and $< v_n >$ are cauchy sequences. Again since $(X, d_1)$, $(Y, d_2)$, $(Z, d_3)$, $(U, d_4)$ and $(V, d_5)$ are complete metric spaces, we get

Taking $x = x_n$ and $y = y_n$ in (1) we get 

Letting $n \to \infty$ we get 


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\[ d_i(PQRS\beta, \alpha) \leq 0 \]

From which it follows that

\[ PQRS\beta = \alpha. \]

In a similar way, using the inequalities (2), (3), (4) and (5) it can be shown that

\[ QRST\alpha = \xi, \quad RSTP\xi = \delta, \quad STPQ\delta = \gamma, \quad TPQR\gamma = \beta. \]

Taking \( z = S\beta \) and \( y = y_n \) in (2) we get

\[ d_d(TPQRS\beta, y_{n+1}) = d_d(TPQRS\beta, TPQRSy_n) \leq \frac{f_2(y_n, S\beta)}{g_2(y_n, S\beta)} \]

\[ = c \max \{ d_d(y_n, y_{n+1})d_d(PQRS\beta, x_n); d_d(y_n, y_{n+1})d_d(QRS\beta, v_n); d_d(y_n, y_{n+1})d_d(RS\beta, u_n); \]

\[ \max \{ d_d(y_n, y_{n+1})d_d(S\beta, TPQRS\beta); d_d(y_n, TPQRS\beta)d_d(S\beta, z_n) \} \]

Letting \( n \to \infty \) and since \( PQRS\beta = \alpha \) we get

\[ d_d(TPQRS\beta, \beta) \leq \frac{c d_d(\beta, TPQRS\beta)d_d(S\beta, \gamma)}{\max \{ d_d(\beta, TPQRS\beta); d_d(\gamma, ST\alpha) \}} \]

Here two Cases arise:

Case (i) If \( \max \{ d_d(\beta, TPQRS\beta); d_d(\gamma, ST\alpha) \} = d_d(\beta, TPQRS\beta) \) we have

\[ d_d(TPQRS\beta, \beta) = d_d(T\alpha, \beta) \leq \frac{c d_d(\beta, TPQRS\beta)d_d(S\beta, \gamma)}{d_d(S\beta, TPQRS\beta)} \]

\[ = c d_d(S\beta, \gamma). \]

Case (ii) If \( \max \{ d_d(\beta, TPQRS\beta); d_d(\gamma, ST\alpha) \} = d_d(\gamma, ST\alpha) \) and \( d_d(\beta, TPQRS\beta) \neq 0 \), we have

\[ d_d(TPQRS\beta, \beta) = d_d(T\alpha, \beta) \leq \frac{c d_d(\beta, TPQRS\beta)d_d(S\beta, \gamma)}{d_d(\gamma, ST\alpha)} \]

\[ = c d_d(\beta, TPQRS\beta)d_d(S\beta, \gamma) \]

\[ = c d_d(S\beta, \gamma) \]

Thus in both cases we have

\[ d_d(TPQRS\beta, \beta) = d_d(T\alpha, \beta) \leq c d_d(S\beta, \gamma) \]

(11)

Taking \( z = z_n \) and \( u = R\gamma \) in (3) we get

\[ d_d(STPQR\gamma, z_{n+1}) = d_d(STPQR\gamma, STPQRz_n) \leq \frac{f_3(z_n, R\gamma)}{g_3(z_n, R\gamma)} \]

\[ = c \max \{ d_d(z_n, z_{n+1})d_d(TPQR\gamma, y_{n+1}); d_d(z_n, z_{n+1})d_d(PQR\gamma, x_n); d_d(z_n, z_{n+1})d_d(QR\gamma, v_n); \]

\[ \max \{ d_d(z_n, STPQR\gamma); d_d(z_n, STPQR\gamma)d_d(R\gamma, u_n) \} \]

Letting \( n \to \infty \) and since \( TPQR\gamma = \beta \) we get

\[ d_d(STPQR\gamma, \gamma) = d_d(S\beta, \gamma) \leq \frac{c d_d(\gamma, STPQR\gamma)d_d(R\gamma, \delta)}{\max \{ d_d(\gamma, STPQR\gamma); d_d(\delta, RS\beta) \}} \]

\[ = c d_d(R\gamma, \delta) \]
Thus as above we get the following inequality
\[ d_3(\text{STPQR} \gamma, \gamma) = d_3(\text{SB}, \gamma) \leq c \cdot d_4(R\gamma, \delta) \]  
(12)

In the similar way using inequalities (4), (5) and (1) we get
\[ d_4(\text{RSTP} \delta, \delta) = d_4(R\gamma, \delta) \leq c \cdot d_4(Q\delta, \xi) \]  
(13)
\[ d_3(\text{QRSTP} \xi, \xi) = d_3(Q\delta, \xi) \leq c \cdot d_4(P\xi, \alpha) \]  
(14)
\[ d_4(\text{PQRST} \alpha, \alpha) = d_4(P\xi, \alpha) \leq c \cdot d_4(T\alpha, \beta) \]  
(15)

Using (11), (12), (13), (14) and (15) we obtain
\[ d_3(\text{TPQRS} \beta, \beta) = d_3(T\alpha, \beta) \leq c \cdot d_3(\text{SB}, \gamma) \leq c^2 \cdot d_4(R\gamma, \delta) \leq c^3 \cdot d_3(Q\delta, \xi) \leq c^4 \cdot d_4(P\xi, \alpha) \leq c^5 \cdot d_4(T\alpha, \beta) = c^5 \cdot d_3(\text{TPQRS} \beta, \beta). \]

From which it follows that
\[ \text{TPQRS} \beta = \beta; \ T\alpha = \beta; \ S\beta = \gamma; \ R\gamma = \delta; \ Q\delta = \xi; \ P\xi = \alpha \] since 0 ≤ c < 1.

Similarly we can show that
\[ \text{STPQR} \gamma = \gamma; \ RSTPQ\delta = \delta; \ QRSTP \xi = \xi; \ PQRST \alpha = \alpha; \ TPQRS \beta = \beta. \]
i.e. \( \alpha, \beta, \gamma, \delta \) and \( \xi \) are fixed points of \( \text{PQRST}, \ \text{TPQRS}, \ \text{STPQR}, \ \text{RSTPQ} \) and \( \text{QRSTP} \) respectively.

Now we show that these fixed points are unique. Let us assume that \( \alpha' \) be the another fixed point of \( \text{PQRST} \).

Using the inequality (1) for \( y = T\alpha \) and \( x = \alpha' \) we get
\[ d_3(\alpha', \alpha') = d_3(\text{PQRST} \alpha, \text{PQRST} \alpha') \leq c \cdot \frac{f_1(\alpha', T\alpha)}{g_1(\alpha', T\alpha)} \]
\[ = c \cdot \max \{ d_1(\alpha', \alpha')d_2(Q\text{QRST} \alpha, \text{QRST} \alpha'); d_3(\alpha', \alpha')d_4(\text{RST} \alpha, \text{RST} \alpha'); d_4(\alpha', \alpha)d_3(T\alpha, T\alpha') \} \]
\[ \max \{ d_1(\alpha', \alpha')d_2(T\alpha, T\alpha') \} \]

Here two cases arise;

Case (i) If \( \max \{ d_1(\alpha', \alpha); d_2(T\alpha, T\alpha) \} = d_2(T\alpha, T\alpha) \) then we get
\[ d_3(\alpha', \alpha') \leq c \cdot d_3(T\alpha, T\alpha') \]
which gives \( \alpha' = \alpha \).

Case (ii) If \( \max \{ d_1(\alpha', \alpha); d_2(T\alpha, T\alpha) \} = d_1(\alpha', \alpha) \) then we get
\[ d_3(\alpha', \alpha') \leq c \cdot d_3(T\alpha, T\alpha') \]  
(16)

Now taking \( z = \text{ST} \alpha \) and \( y = T\alpha' \) in equation (2) we get
\[ d_3(T\alpha, T\alpha') = d_3(\text{TPQRST} \alpha, \text{TPQRST} \alpha') \leq c \cdot \frac{f_2(T\alpha', \text{ST} \alpha)}{g_2(T\alpha', \text{ST} \alpha)} \]
\[ = c \cdot \max \{ d_2(T\alpha, T\alpha')d_3(\alpha', \alpha'); d_3(T\alpha, T\alpha')d_4(\text{QRST} \alpha, \text{QRST} \alpha'); d_4(T\alpha, T\alpha')d_3(T\alpha, T\alpha') \} \]
\[ \max \{ d_2(T\alpha, T\alpha); d_3(T\alpha, T\alpha'); d_3(\text{ST} \alpha, \text{ST} \alpha') \} \]

As discussed above we get
\[ d_3(T\alpha, T\alpha') \leq c \cdot d_3(\text{ST} \alpha, \text{ST} \alpha') \]  
(17)

Similarly taking \( u = \text{RST} \alpha \) and \( x = \text{ST} \alpha' \) in equation (3) we get
\[ d_3(\text{ST} \alpha, \text{ST} \alpha') \leq c \cdot d_4(\text{RST} \alpha, \text{RST} \alpha') \]  
(18)

Taking \( v = \text{QRST} \alpha \) and \( u = \text{RST} \alpha' \) in equation (4) we get
\[ d_3(\text{RST} \alpha, \text{RST} \alpha') \leq c \cdot d_4(\text{QRST} \alpha, \text{QRST} \alpha') \]  
(19)

Taking \( x = \text{PQRST} \alpha \) and \( v = \text{QRST} \alpha' \) in equation (5) we get
\[ d_3(\text{QRST} \alpha, \text{QRST} \alpha') \leq c \cdot d_4(\text{PQRST} \alpha, \text{PQRST} \alpha') \]
\[ = c \cdot d_1(\alpha, \alpha') \]  
(20)

Using equation (16), (17), (18), (19) and (20) we get
d₁(α, α') ≤ c d₂(Tα, Tα') ≤ c² d₃(STα, STα') ≤ c³ d₄(RSTα, RSTα') ≤ c⁴ d₅(QRSTα, QRSTα') ≤ c⁵ d₁(α, α')

Which is impossible since 0 ≤ c < 1.

Thus d₁(α, α') = 0 i.e. α = α' i.e. α is a unique fixed point of PQRST. In the same way we can prove that β, γ, δ and ξ be the fixed point of TPQRS, STPQR, RSTPQ and QRSTP respectively.

This completes the proof of the theorem.

REFERENCES