On Degree of Approximation of Conjugate Series of Fourier Series by Product Means \((E, q)A\)

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ABSTRACT
In this paper a theorem on degree of Approximation of a function \(f \in Lip \alpha\) by product summability \((E, q)A\) of conjugate series of Fourier series associated with \(f\).

KEYWORDS: Degree of Approximation, \(Lip \alpha\) class of function, \((E, q)\)-mean, \(A\)-mean, \((E, q)A\)-product mean, conjugate Fourier series, Lebesgue integral.

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INTRODUCTION
Let \(\sum a_n\) be a given infinite series with the sequence of partial sums \(\{s_n\}\). Let \(A = (a_{mn})_{m,n}\) be a matrix. Then the sequence \(-to-sequence transformation

\[ t_n = \sum_{l=0}^{m} a_{ml} s_l, \quad n = 1, 2, \cdots \]

defines the sequence \(\{t_n\}\) of the \(A\)-mean of the sequence \(\{s_n\}\). If

\[ t_n \to s \quad \text{as} \quad n \to \infty, \]

then the series \(\sum a_n\) is said to be \(A\) summable to \(s\).

The conditions for regularity of \(A\)-summability are easily seen to be

(i) \(\sup_{m} \sum_{n=0}^{\infty} |a_{mn}| < H\) where \(H\) is an absolute constant.

(ii) \(\lim_{m \to \infty} a_{mn} = 0\)

(iii) \(\lim_{m \to \infty} \sum_{n=0}^{\infty} a_{mn} = 1\)

The sequence \(-to-sequence transformation, [1]

\[ T_n = \frac{1}{(1 + q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{n-l} s_l \]

defines the sequence \(\{T_n\}\) of the \(A\) mean of the sequence \(\{s_n\}\). If
clearly \((E,q)\) method is regular [1]. Further, the \((E,q)\) transform of the \(A\) transform of \(\{s_n\}\) is defined by

\[
\tau_n = \frac{1}{(1+q)^n} \left[ \sum_{k=0}^{n} \binom{n}{k} q^{-k} t_k \right]
\]

If \(\sum a_n\) is said to be \((E,q)\)-summable to \(s\), then the series \(\sum a_n\) is said to be \((E,q)\)-summable to \(s\).

Let \(f(t)\) be a periodic function with period \(2\pi\), \(L\)-integrable over \((-\pi,\pi)\). The Fourier series associated with \(f\) at any point \(x\) is defined by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos nx + b_n \sin nx \right] \equiv \sum_{n=0}^{\infty} A_n(x)
\]

and the conjugate series of the Fourier series (1.8) is

\[
\sum_{n=1}^{\infty} \left[ a_n \cos nx - b_n \sin nx \right] \equiv \sum_{n=1}^{\infty} B_n(x)
\]

Let \(S_n(f; x)\) be the \(n\)-th partial sum of (1.9).

The \(L_\infty\)-norm of a function \(f : \mathbb{R} \rightarrow \mathbb{R}\) is defined by

\[
\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|
\]

and the \(L_0\)-norm is defined by

\[
\|f\|_0 = \left( \frac{2\pi}{\int_0^1 |f(x)|^\nu dx} \right)^{\frac{1}{\nu}}, \quad \nu \geq 1
\]

The degree of approximation of a function \(f : \mathbb{R} \rightarrow \mathbb{R}\) by a trigonometric polynomial \(P_n(x)\) of degree \(n\) under norm \(\|\cdot\|_\infty\) is defined by [3].

\[
\|P_n - f\|_\infty = \sup_{x \in \mathbb{R}} |P_n(x) - f(x)|
\]

and the degree of approximation \(E_n(f)\) of a function \(f \in L_0\) is given by

\[
E_n(f) = \min_{P_n} \|P_n - f\|_0
\]

This method of approximation is called trigonometric Fourier approximation.

A function \(f \in Lip \alpha\) if

\[
|f(x+t) - f(x)| = O\left(|t|^\alpha\right), 0 < \alpha \leq 1
\]

We use the following notation throughout this paper:

\[
\psi(t) = \frac{1}{2} \left\{ f(x+t) - f(x-t) \right\},
\]

and
Further, the method \((E,q)A\) is assumed to be regular and this case is supposed throughout the paper.

**KNOWN THEOREM**

Dealing with the degree of approximation by the product \((E,q)\) \((C,1)\)-mean of Fourier series, Nigam [2] proved the following theorem.

**Theorem- 2.1:**

If a function \(f\) ,\(2\pi\)-periodic, belonging to class \(Lip\alpha\), then its degree of approximation by \((E,q)\) \((C,1)\) summability mean on its Fourier series \(\sum_{n=0}^{\infty} A_n(t)\) is given by

\[
\|E_n^\alpha C_n^1 - f\|_{\infty} = O\left(\frac{1}{(n+1)^\alpha}\right), \quad 0 < \alpha < 1, \quad \text{where } E_n^\alpha C_n^1 \text{ represents the } (E,q) \text{ transform of } (C,1) \text{ transform of } s_n(f; x).
\]

**MAIN THEOREM**

In this paper, we have proved a theorem on degree of approximation by the product mean \((E,q)A\) of conjugate series of Fourier series of (1.8), we prove:

**Theorem - 3.1:**

If \(f\) is a \(2\pi\)-Periodic function of class \(Lip\alpha\), then degree of approximation by the product \((E,q)A\) summability means on its conjugate series of Fourier series (1.8) is given by

\[
\|\tau_n - f\|_{\infty} = O\left(\frac{1}{(n+1)^\alpha}\right), \quad 0 < \alpha < 1, \quad \text{where } \tau_n \text{ as defined in (1.5)}.
\]

**LEMNAS**

We require the following Lemmas to prove the theorem-3.1.

**Lemma - 4.1:**

\[
\left| K_n(t) \right| = O(n), \quad 0 \leq t \leq \frac{1}{n+1}, \quad \text{where } K_n(t) \text{ is as defined in (1.15)}
\]

**Proof of Lemma- 4.1:**

For \(0 \leq t \leq \frac{1}{n+1}\), we have \(\sin nt \leq nsint\), then

\[
\left| K_n(t) \right| = \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left| \sum_{u=0}^{k} a_{ku} \cos\left(\frac{t}{2}\cos\left(\frac{u+1}{2}\right)t\right) \sin\left(\frac{t}{2}\right) \right|
\]
\[
\left| \frac{1}{\pi (1 + q)^n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \left[ \cos \frac{t}{2} - \cos \nu t \cos \frac{t}{2} + \sin \nu t \sin \frac{t}{2} \right] \right| \\
\leq \frac{1}{\pi (1 + q)^n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \left[ \cos \frac{t}{2} \left( 2 \sin^2 \frac{\nu t}{2} \right) + \sin \nu t \right] \\
\leq \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \left( O(\nu) + O(\nu) \right) \right| \\
\leq \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \right| \\
= \frac{H}{\pi (1 + q)^n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} O(k) \sum_{\nu=0}^{k} a_{k\nu}, \text{ by regularity condition} \\
= O(n).
\]

This proves the lemma.

**Lemma- 4.2:**

\[
\left| \overline{K}_n(t) \right| = O\left( \frac{1}{t} \right), \text{ for } \frac{1}{n+1} \leq t \leq \pi, \text{ where } \overline{K}_n(t) \text{ is as defined in (1.15)}
\]

**Proof of Lemma- 4.2:**

For \( \frac{1}{n+1} \leq t \leq \pi \), we have by Jordan’s lemma, \( \sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi} \).

Then

\[
\left| \overline{K}_n(t) \right| = \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \left[ \cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right) \right] \right| \\
= \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \frac{1}{P_k} \sum_{\nu' \leq \nu} p_{k-\nu} \cos \frac{t}{2} - \cos \frac{\nu t}{2} \cos \frac{t}{2} + \sin \frac{\nu t}{2} \sin \frac{t}{2} \right| \\
\leq \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{-k} \sum_{\nu=0}^{k} a_{k\nu} \frac{1}{P_k} \sum_{\nu' \leq \nu} p_{k-\nu} \cos \frac{t}{2} - \cos \frac{\nu t}{2} \cos \frac{t}{2} + \sin \frac{\nu t}{2} \sin \frac{t}{2} \right|. 
\]
\[
\leq \frac{1}{\pi (1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} \frac{\pi}{2\nu} a_{k\nu} \cos \frac{t}{2} \left( 2\sin^2 \frac{\nu t}{2} + \sin \frac{\nu t}{2} \sin \frac{t}{2} \right) \right\}
\]
\[
\leq \frac{\pi}{2\pi (1+q)^n t} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} a_{k\nu} \right\}
\]
\[
= \frac{1}{2(1+q)^n t} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} a_{k\nu} \right\}.
\]
\[
\leq \frac{H}{2(1+q)^n t} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \text{ by regularity condition}
\]
\[
= O\left( \frac{1}{t} \right). \tag{5.1}
\]

This proves the lemma.

**PROOF OF THEOREM - 3.1**

Using Riemann –Lebesgue theorem, we have for the n-th partial sum \( s_n(f; x) \) of the conjugate Fourier series (1.8) of,

\[
\overline{s}_n(f; x) - f(x) = \frac{2}{\pi} \int_0^\pi \psi(t) K_n(t) \ dt,
\]

following Titechmarch [3]the \( A \) – transform of \( s_n(f; x) \) using (1.1) is given by

\[
t_n - f(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \sum_{k=0}^{n} a_{nk} \frac{\cos \frac{1}{2} \left( n + \frac{1}{2} \right) t}{2\sin \left( \frac{t}{2} \right)} dt,
\]

denoting the \( (E, q)A \) transform of \( s_n(f; x) \) by \( \tau_n \), we have

\[
\| F_n - f \| = \frac{2}{\pi (1+q)^n} \int_0^\pi \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} \frac{\cos \frac{1}{2} \left( \nu + \frac{1}{2} \right) t}{2\sin \left( \frac{t}{2} \right)} \right\} dt
\]
\[
= \int_0^\pi \psi(t) K_n(t) dt
\]
\[
= \left\{ \int_0^{\frac{1}{n+1}} + \int_0^{\frac{\pi}{n+1}} \right\} \psi(t) K_n(t) \ dt
\]

\[
(5.1) \quad = I_1 + I_2, \text{ say}
\]

Now
\[ |I_1| = \frac{2}{\pi (1+q)^n} \left| \int_0^{\frac{1}{n+1}} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} a_{\nu} \frac{\cos \frac{\nu}{2} - \cos \left( \frac{\nu + 1}{2} \right)}{2 \sin \frac{\nu}{2}} \right\} dt \right| \]

\[ \leq O(n) \int_0^{\frac{1}{n+1}} |\psi(t)| dt , \text{ using Lemma 4.1} \]

\[ = O(n) \int_0^{\frac{1}{n+1}} |t^\alpha| dt \]

\[ = O(n) \left[ t^{\alpha+1} \right]_0^{\frac{1}{n+1}} \]

\[ = O(n) \left[ \frac{1}{(\alpha+1)(n+1)^{\alpha+1}} \right] . \]

\[ = O \left[ \frac{1}{(n+1)^\alpha} \right] \]

Next

\[ |I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)| \left| \overline{K_n}(t) \right| dt \]

\[ = \int_{\frac{1}{n+1}}^{\pi} |\psi(t)| O \left( \frac{1}{t} \right) dt , \text{ using Lemma 4.2} \]

\[ = \int_{\frac{1}{n+1}}^{\pi} |t^\alpha| O \left( \frac{1}{t} \right) dt \]

\[ = \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1} dt \]

\[ = O \left( \frac{1}{(n+1)^\alpha} \right) \]

Then from (5.2) and (5.3), we have

\[ |\tau_n - f(x)| = O \left( \frac{1}{(n+1)^\alpha} \right) , \text{ for } 0 < \alpha < 1. \]
Hence, $\|r_n - f(x)\|_{\infty} = \sup_{-\pi < x < \pi} |r_n - f(x)| = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1$.

This completes the proof of the theorem 3.1.

REFERENCES