# Spectra Corresponding to Sierpinski Graphs 

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#### Abstract

The paper stems from an attempt to investigate a somewhat incomprehensible pattern(i.e. fractal looklike) which suffice for the existence of Sierpinski graph. Graphs of "Sierpinski" type appear naturally in many different areas of mathematics as well as in several other fields. There are a wide variety of graph matrix representations. Among these are the adjacency matrix, incidence matrix, circuit matrix, Laplacian matrix and Signless Laplacian matrix. Here we introduces new type of Sierpinski graph, said Sierpinski Eulerian graph. Spectra of Sierpinski graph can also be derived by studying eigenvalues. The choice of matrix representation clearly has a large effect on the suitability of spectrum in a number of pattern recognition tasks. The objectives of this research are to find compare spectra of graph matrix representation.


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## INTRODUCTION

This is not a sudden discovery but a gradual realisation as recently as few years earlier we do not have certain data that made interpretation of the self similar structures that now we calling a Fractal. [1]Graph theory is a branch of mathematics started by Euler as early as 1736. It took a hundred years before the second important contribution of Kirchhoff had been made for the analysis of electrical networks. [1]There are many physical applications whose performance depends not only on the characteristics of their components but also on their relative location. On the other hand, if the location of a member is changed, the properties of the structure will again be different. Therefore, the connectivity (topology) of the structure influences the performance of the entire structure. Hence, it is important to represent a system so that its topology can clearly be understood. [3,8,16,20] Some of the uses of the theory of graphs in the context of civil engineering are as follows. A graph can be a model of a structure, a hydraulic network, a traffic network, a transportation system, a construction system or a resource allocation system. These are only some of such models, and the applications of graph theory are much extensive.
[12,13,15] Sierpinski's Triangle is one of the most famous examples of a fractal although we should note that Benoit Mandelbrot first used the term fractal in 1975, almost sixty years after Sierpinski created his famous triangle.
The generalised Sierpinski graph, as per the above definition of the Sierpinski graphs $S(n, k)$. The vertex set of $S(n, k)$ consists of all $n$-tuples of the integers (for every
$n \geq 1$ and $k \geq 1) \quad$ i.e. $\quad V(S(n, k))=\{1,2,3 \ldots \ldots . ., k\}^{n}$.Two different vertices
$u=\left(u_{1}, u_{2}, \ldots, u_{n}\right)$ and $v=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ are adjacent if and only if there exists an $h \in\{1,2, \ldots, n\}$ such that
(i) $u_{t}=v_{t}$, for $t=1,2, \ldots \ldots, h-1$;
(ii) $u_{h} \neq v_{h}$; and
(iii) $u_{t}=v_{h}$ and $v_{t}=u_{h}$ for $t=h+1, \ldots \ldots, n$.

Trail that visits every edge of the graph once and only once is called Eulerian trail. Starting and ending vertices are different from the one on which it began. A graph of this kind is said to be traversable. An Eulerian circuit is an Eulerian trail that is a circuit i.e., it begins and ends on the same vertex. A graph is called Eulerian when it contains an Eulerian circuit. A digraph in which the in-degree equals the out-degree at each vertex.
Theorem: An undirected graph has at least one Euler path if and only if it is connected and has two or zero vertices of odd degree.
Theorem: An undirected graph has an Euler circuit if and only if it is connected and has zero vertices of odd degree.
Proposition: Sierpinski's Gasket has an Euler circuit if and only if it is has two or zero vertices of odd degree.
Proposition: Sierpinski's Gasket is Eulerian if and only if its vertices are all of even degree. Proof:
Case 1(Eulerian): Suppose $G$ be a Sierpinski Graph is Eulerian, then G has an Eulerian trail which begins and ends at " $a$ ". If traverse along the trail then each and every time traverse a vertex having two edges. It is necessary condition that starting and ending nodes are same and each and every vertices must contain even degree (deg(v)) of vertices.
Case 2( not Eulerian): Suppose $G$ be a Sierpinski Graph is not Eulerian, then G has not Eulerian trail which begins at " $a_{1}$ " but does not ends at " $a_{1}$ ". If traverse along the trail then each and every time traverse a vertex having two odd vertices or even vertices but above figure does not satisfy the Eulerian condition. Since each vertex in the middle of the trail is associated with three edges ( $G$ can not have just one odd vertex).

## Eigenvalues of a graph

Let $A$ be the adjacency matrix of the graph $\Gamma$ of order $N$. Let $I$ be the identity matrix of order $N$, and let $\lambda$ be a scalar. Then the determinant $|\boldsymbol{A} \boldsymbol{\lambda} \boldsymbol{I}|$ which is an ordinary polynomial in $\lambda$ of $N$-th degree with scalar coefficients, is called the characteristic polynomial of $\Gamma$. The roots of the equation $|A-\lambda I|=0$ are called the eigenvalues of the graph $\Gamma$ (also of the matrix $A$ ). The set of eigenvalues is called the spectrum of the graph $\Gamma$. The multiplicity of an eigenvalue $\lambda$ is called the algebraic multiplicity of $\lambda$. The equation $\boldsymbol{A} \boldsymbol{u}=\boldsymbol{\lambda} \boldsymbol{u}$ is called an eigenvalue equation. A nonzero solution $u$ of the equation is called an eigenvector or eigenfunction for the eigenvalue $\lambda$. The vector space constructed from the set of eigenvectors corresponding to a particular eigenvalue $\lambda$ is called the eigenspace of $\lambda$. The dimension of the eigenspace of an eigenvalue $\lambda$ is the geometric multiplicity of $\lambda$. For a symmetric matrix, the geometric and algebraic multiplicities of an eigenvalue are equal.

## Laplacian Matrix:-

We consider graphs which has no loops or parallel edges, unless stated otherwise. The adjacency matrix $A(G)$ of $G$ is an $n \times n$ matrix with its rows and columns indexed by $V(G)$ and with the $(i, j)$ - entry equal to 1 if vertices $i, j$ are adjacent (i.e., joined by an edge) O(zero) otherwise. Thus $A(G)$ is a symmetric matrix with its $i$-th row (or column) sum equal to $d_{i}(G)$, which by definition is the degree of the vertex $i, i=1,2, \ldots \ldots, n$. Let $D(G)$ denote the $n \times n$ diagonal matrix, $i-$ th diagonal entry is $d_{i}(G), i=1,2, \ldots \ldots, n$.
The Laplacian matrix of $G$, denoted by $L(G)$, is simply the matrix $D(G)-A(G)$.

## Signless Laplacian Matrix:-

The Signless Laplacian matrix of $G$, denoted by $L(G)$, is simply the matrix $D(G)+A(G)$.
Theorem: Let $G$ be a graph on n vertices with vertex degrees $d_{1}, d_{2}, \ldots . ., d_{n}$ and largest $Q$ eigenvalue $q_{1}$. Then $2 \min d_{i} \leq q_{1} \leq 2 \max d_{i}$. For a connected graph $G$, equality holds in either of these in equalities if and only if $G$ is regular.

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Theorem: Let $G$ be a graph on $n$ vertices with vertex degrees $d_{1}, d_{2}, \ldots ., d_{n}$ and largest $Q$ eigenvalue $q_{1}$.Then $\min \left(d_{i}+d_{j}\right) \leq q_{1} \leq \max \left(d_{i}+d_{j}\right)$, where $(i, j)$ runs over all pairs of adjacent vertices of $G$. For a connected graph $G$, equality holds in either of these inequalities if and only if $G$ is regular or semi-regular bipartite.
Proof: The line graph $L(G)$ of $G$ has largest eigenvalue $q_{1}-2$. Consider an edge $u$ of $G$ which joins vertices $i$ and $j$. The vertex $u$ of $L(G)$ has degree $d_{i}+d_{j}-2$. Hence, $\min \left(d_{i}+d_{j}-2\right) \leq q_{1}-2 \leq \max \left(d_{i}+d_{j}-2\right)$, which proves the theorem.
Lemma: Let $p(x)$ be a given polynomial. If $\lambda$ is an eigenvalue of $A$, while $x$ is an associated eigenvector, then $p(\lambda)$ is an eigenvalue of the matrix $p(A)$ and $x$ is an eigenvector of $p(A)$ associated with $p(\lambda)$. The characteristic polynomial of $A$ is defined by
$\chi_{A}(t)=\operatorname{det}(t I-A)$
Lemma: Let $A$ be a $n \times n$-matrix with eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$. Then

$$
\operatorname{tr}(A)=\sum_{i=1}^{n} \lambda_{i}
$$

Lemma: Let $A$ be a symmetric real matrix. Suppose $v$ and $w$ are eigenvectors of $A$ associated with the eigenvalues $\lambda$ and $\mu$ respectively. If $\lambda \neq \mu$ then $v \perp w$, i.e. $v$ and $w$ are orthogonal.
Proposition: The least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.
Proof: Let $x^{T}=\left(x_{1}, \ldots \ldots, x_{n}\right)$. For a non-zero vector $x$ we have $Q x=0$ if and only if $R^{T} x=0$. The later holds if and only if $x_{i}=-x_{j}$ for every edge, i.e. if and only if $G$ is bipartite. Since the graph is connected, $x$ is determined up to a scalar multiple by the value of its coordinate corresponding to any fixed vertex $i$.
Theorem: (Spectral Theorem) Let $A$ be a $n \times n$ symmetric real matrix. Then there are $n$ pairwise orthogonal (real) eigenvectors $v_{i}$ of $A$ associated with real eigenvalues of $A$.
Consider $\lambda_{1}(A) \leq \ldots \leq \lambda_{\mathrm{n}}(A)$ are eigenvalues of a symmetric matrix $A$. Some of these eigenvalues can be equal; we say that those eigenvalues have multiplicity greater than 1. Thus we denote the spectrum of $A$ also in the form $\bar{\lambda}_{1}^{\left[m_{1}\right]}, \ldots \ldots, \bar{\lambda}_{2}{ }^{\left[m_{2}\right]}$, where $\bar{\lambda}_{i}$ is an eigenvalue with multiplicity $m_{i}$.
Lemma[17,21]: Let $G$ be a graph on $n$ vertices.
i) The maximum eigenvalue of $G$ lies between the average and the maximum degree of $G$, i.e.

$$
\bar{d} \leq \lambda_{n} \leq \Delta
$$

ii) The range of all the eigenvalues of a graph is $-\Delta \leq \lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n} \leq \Delta$.

Definiton: (Laplacian eigenvalues) The eigenvalues of $L(G)$ are called the Laplacian eigenvalues. The set of all the Laplacian eigenvalues are called the (Laplacian) spectrum of the graph $G$.
Lemma[8]: Let $G$ be a graph on $n$ vertices with Laplacian eigenvalues $\lambda_{1}=0 \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$ and maximum degree $\Delta$. Then $0 \leq \lambda_{i} \leq 2 \Delta$ and $\lambda_{n} \geq \Delta$.
Proof: All eigenvalues are nonnegative by positive semidefinite matrices.
Let $u$ be an eigenvector corresponding to the eigenvalue $\lambda$, and let $u_{j}$ denote the entry with the largest absolute value. We have
$|\lambda| u_{j}\left|=\left|\lambda u_{j}\right|=\left|d_{j} u_{j}-\sum_{i \sim j} u_{i}\right| \leq d_{j}\right| u_{j}\left|+\sum_{i \sim j}\right| u_{i}\left|\leq 2 d_{j}\right| u_{j}|\leq 2 \Delta| u_{j} \mid$.
Thus, we have $|\lambda| \leq 2 \Delta$ as required.
Let $j$ be the vertex with maximal degree, i.e. $d_{j}=\Delta$. We define the characteristic vector $x$ :
$x_{i}= \begin{cases}1, & \text { if } i=j ; \\ 0, & \text { otherwise } .\end{cases}$
Now, the desired inequality follows:

$$
\lambda_{n}=\max _{\widetilde{x} \neq 0} \frac{\tilde{x}^{T} \tilde{x}}{\widetilde{x}^{T} \widetilde{x}} \geq \frac{x^{T} L x}{x^{T} x}=\frac{\sum_{\{u, v\} \in E}\left(x_{u}-x_{v}\right)^{2}}{1}=\Delta
$$

## RESULT AND DISCUSSION

The Laplacian and Signless laplacian eigenvalues of the representation matrices $L$ and $\bar{L}$ of Siepinski graph and Sierpinski Eulerian graph. The eigenvalue spectra become more comparable via the proposed notations $\lambda 1, \lambda 2$ of Sierpinski graph and Sierpinski Eulerian respectively.



Results for Laplacian matrix of Sierpinski and Sierpinski Eulerian graph dataset. (b) eigenvalues $\lambda 1$ of Laplacian matrix of Sierpinski Graph and (e) eigenvalues $\lambda 2$ of Laplacian matrix of Sierpinski Eulerian graph.


Results for Signless Laplacian matrix of Sierpinski and Sierpinski Eulerian graph dataset: (c) eigenvalues $\lambda 1$ of Signless Laplacian matrix of Sierpinski Graph and (f) eigenvalues $\lambda 2$ of Signless Laplacian matrix of Sierpinski Eulerian graph.

Adding connected components. We now turn to a specific manipulation of graphs addition of connected components which allows us to order the spectra of the graphs observed in the above graphs. As above we see that we can obtain Sierpinski Eulerian graph from Sierpinski graph by adding a edges between pair of odd degree vertices and that we can obtain Sierpinski Eulerian graph by adding the connected components [14]. In general, we can obtain a graph in $\mathrm{C}_{\mathrm{j}, \mathrm{k}}$ from a graph in $\mathrm{C}_{\mathrm{j}+1, \mathrm{k}}$, for all $\mathrm{j} \leq \mathrm{k}-1, \mathrm{k} \in \mathrm{N}$, by adding to the graph one or more connected components in which all vertices are of degree greater or equal to j and smaller or equal to k , with at least one vertex attaining degree j .
The above data results that the Signless Laplacian-spectrum and Laplacian-spectrum is used to encode graphs and has more representational power. Also, shows that signless Laplacian eigenvalues and Laplacian eigenvalues have better use, as they have stronger characterization properties.
Since the Signless Laplacian spectra perform better also in comparison to spectra of other commonly used graph matrices (Laplacian, the adjacency matrix)as expressed in [7] that, the signless Laplacian seems to be the most convenient for use in studying graph properties.

## CONCLUSION

Spectral graph theory provides another approach to the problem of graph similarity $[2,3,4]$. This approach is based on a branch of mathematics that is concerned with characterising the structural properties of graphs [6]. There are many results in the mathematical literature on spectral characterizations of particular classes of graphs, see $[3,8,16,19]$.

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Here, we have compared the spectra of the two graph representation matrices the Laplacian matrices and the Signless Laplacian matrices and found differences in the spectra corresponding to Sierpinski graphs. Signless Laplacian matrices have more representative value as compare to Laplacian matrices. As a result of this work, we hope to have increased awareness about the importance of the choice of representation of matrix for graph signal processing applications and other fields of communications [3]. However, these results hardly could be applied to graphs which appear in applications to computer science, science, mathematics and other aspects of NP problems.

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