On The Homogeneous Bi-quadratic Equation with Five Unknowns

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ABSTRACT

The Bi-quadratic Equation with 5 unknown given by \( x^4 - y^4 = 5(z^2 - w^2)R^2 \) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Key words: Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

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INTRODUCTION

Bi-quadratic Diophantine Equations, homogeneous and non-homogeneous, have aroused the interest of numerous Mathematicians since ambiguity as can be seen from [1-7]. In the context one may refer [8-20] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another quadratic equation in 5 unknowns represented by \( x^4 - y^4 = 5(z^2 - w^2)R^2 \). A few interesting relations between the solutions and special polygonal numbers are presented.

NOTATIONS USED

- \( t_{m,n} \) - Polygonal number of rank \( n \) with size \( m \).
- \( P^n_m \) - Pyramidal number of rank \( n \) with size \( m \).
- \( cP^n_{m,n} \) - Centered polygonal number of rank \( n \) with size \( m \).
- \( g^n_a \) - Gnomonic number of rank \( a \)
- \( s^n_o \) - Stella octangular number of rank \( n \)
- \( s^n \) - Star number of rank \( n \)
- \( p^n_r \) - Pronic number of rank \( n \)
- \( p^n_t \) - Pentatope number of rank \( n \)
- \( CP^n_{m,n} \) - Centered pyramidal number of rank \( n \) with size \( m \)

METHOD OF ANALYSIS

The Diophantine equation representing the bi-quadratic equation with five unknowns under consideration is

\[ x^4 - y^4 = 5(z^2 - w^2)R^2 \] (1)

The substitution of the linear transformations

\[ x = u + v, \ y = u - v, \ z = 2u + v, \ w = 2u - v \] (2)
in (1) leads to \[ u^2 + v^2 = 5R^2 \] \tag{3}

Different patterns of solutions of (1) are presented below.

**Pattern - 1**

Assume \( R = a^2 + b^2 \) where \( a \) and \( b \) are non-zero distinct integers. \tag{4}

Write 5 as \( 5 = (2 + i)(2 - i) \) \tag{5}

Using (4) & (5) in (3) and employing the method of factorization, define

\[ u + iv = (2 + i)(a + ib)^2 \]

Equating the real and imaginary parts, we get

\[ u = u(a, b) = 2a^2 - 2b^2 - 2ab \]
\[ v = v(a, b) = a^2 - b^2 + 4ab \]

Hence in view of (2) the corresponding solutions of (1) are

\[ x = x(a, b) = 3a^2 - 3b^2 + 2ab \]
\[ y = y(a, b) = a^2 - b^2 - 6ab \]
\[ z = z(a, b) = 5a^2 - 5b^2 \]
\[ w = w(a, b) = 3a^2 - 3b^2 - 8ab \]
\[ R = R(a, b) = a^2 + b^2 \]

A few interesting properties observed are as follows:

1. \( x(a, a(a+1)) - 3y(a, a(a+1)) = 40p_5^5 \)
2. \( z(a, b) - 5y(a, b) \equiv 0 \pmod{30} \)
3. \( x(a, (a+1)(a+2)) - w(a, (a+1)(a+2)) = 60p_5^3 \)
4. \( z(a, b) + R(a, b) = Nastynumber - t_{4,2b} \)

5. Each of the following represents a nasty number:
   - \( 3\{y(a, 2a^2 - 1) + R(a, 2a^2 - 1) + 6SO_a\} \)
   - \( 75R(a, b) + 15z(a, b) \)
   - \( z(a, a) - y(a, a) \)

**Pattern-2:**

Instead of (4) write 5 as

\[ 5 = (1 + 2i)(1 - 2i) \] \tag{6}

Following a similar procedure as in pattern-1, the solutions for (3) are as follows

\[
\begin{aligned}
   u &= u(a, b) = a^2 - b^2 - 4ab \\
   v &= v(a, b) = 2a^2 - 2b^2 + 2ab
\end{aligned}
\] \tag{7}
In view of (2) and (7) the solutions of (1) are obtained as
\[ x = x(a, b) = 3a^2 - 3b^2 - 2ab \]
\[ y = y(a, b) = -a^2 + b^2 - 6ab \]
\[ z = z(a, b) = 4a^2 - 4b^2 - 6ab \]
\[ w = w(a, b) = -10ab \]
\[ R = R(a, b) = a^2 + b^2 \]

**Properties:**
1. \( x(a, b) + 3y(a, b) = 2w(a, b) \equiv 0 \pmod{20} \)
2. \(-z(a, 2a^2 + 1) - 4y(a, 2a^2 + 1) = 90(Oh_a) \)
3. \( x(a, b) - y(a, b) - z(a, b) + w(a, b) = 0 \)
4. \( x(a, a^2) + y(a, a^2) = 2(t_{4, a} - t_{4, a^2}) + CP_{6, 2a} \)
5. Each of the following represents a nasty number:
   - \( 3\{ -y(a, a) - R(a, a) - 2t_{4, a} \} \)
   - \( -y(a, a) \) and \(-z(a, a) \)

**Pattern-3:**
In addition to (4) and (6), write 5 as
\[ 5 = \frac{1}{25}(11 + 2i)(11 - 2i) \]

Following the procedure as in pattern-2, the solutions for (3) are as follows
\[ u = u(a, b) = \frac{1}{5}(11a^2 - 11b^2 - 4ab) \]
\[ v = v(a, b) = \frac{1}{5}(2a^2 - 2b^2 + 22ab) \]

Hence the corresponding solutions of (1) are
\[ x = x(a, b) = \frac{1}{5}(13a^2 - 13b^2 + 18ab) \]
\[ y = y(a, b) = \frac{1}{5}(9a^2 - 9b^2 - 26ab) \]
\[ z = z(a, b) = \frac{1}{5}(24a^2 - 24b^2 + 14ab) \]
\[ w = w(a, b) = \frac{1}{5}(20a^2 - 20b^2 - 30ab) \]

As our interest on finding integer solutions, we choose a and b suitably so that the values of \( x, y, z, w \) are integers.

**Illustration I:**
Let \( a = 5A \) and \( b = 5B \)

Thus the corresponding solutions of (1) are
In view of (2) and (11), the solutions of (1) are obtained as

\[
\begin{align*}
x & = x(a, b) = 15a^2 + b^2 + 10ab \\
y & = y(a, b) = -5a^2 - 3b^2 - 10ab \\
z & = z(a, b) = 20a^2 + 10ab \\
w & = w(a, b) = -4b^2 - 10ab \\
R & = R(a, b) = 5a^2 + b^2 + 4ab
\end{align*}
\]
Properties:

1. $z(a, b) + w(a, b) = 2(x(a, b) + y(a, b)) \equiv 0 \pmod{4}$
2. $x(a, -1) - R(a, -1) = 4t_{a,b}$
3. $x(a, b) - y(a, b) - z(a, b) - t_{a,b} \equiv 0 \pmod{10}$
4. Each of the following represents a nasty number:
   - $3\{x(a, a) + y(a, a)\}$
   - $x(a, a) + y(a, a) + z(a, a) + w(a, a)$

Pattern-5:

Instead of (9), write $1 = \frac{1}{4}(\sqrt{5} + 1)(\sqrt{5} - 1)$

Following the same procedure as in pattern-4, the solutions for (3) are as follows

$$R = R(a, b) = \frac{1}{2}(5a^2 + b^2 + 2ab)$$
$$v = v(a, b) = \frac{1}{2}(5a^2 + b^2 + 10ab)$$

(12)

In view of (2) and (12), the solutions of (1) are

$$x = x(a, b) = \frac{1}{2}(15a^2 - b^2 + 10ab)$$
$$y = y(a, b) = \frac{1}{2}(5a^2 - 3b^2 - 10ab)$$
$$z = z(a, b) = \frac{1}{2}(25a^2 - 3b^2 + 10ab)$$
$$w = w(a, b) = \frac{1}{2}(15a^2 - 5b^2 - 10ab)$$
$$R = R(a, b) = \frac{1}{2}(5a^2 + b^2 + 2ab)$$

The values of $x, y, z, w$ and $R$ are integers when both $a$ and $b$ are of the same parity.

Case- I:

Consider $a = 2A$ and $b = 2B$

Thus the corresponding solutions of (1) are

$$x = x(A, B) = 30A^2 - 2B^2 + 20AB$$
$$y = y(A, B) = 10A^2 - 6B^2 - 20AB$$
$$z = z(A, B) = 50A^2 - 6B^2 + 20AB$$
$$w = w(A, B) = 30A^2 - 10B^2 - 20AB$$
$$R = R(A, B) = 10A^2 + 2B^2 + 4AB$$

Case- II:

Put $a = 2A + 1$ and $b = 2B + 1$
Hence the corresponding solutions of (1) are

\[ x = x(A, B) = 30A^2 - 2B^2 + 40A + 8B + 20AB + 12 \]
\[ y = y(A, B) = 10A^2 - 6B^2 - 16B - 20AB - 4 \]
\[ z = z(A, B) = 50A^2 - 6B^2 + 60A + 4B + 20AB + 16 \]
\[ w = w(A, B) = 30A^2 - 10B^2 + 20A - 20B - 20AB \]
\[ R = R(A, B) = 10A^2 + 2B^2 + 12A + 4B + 4AB + 4 \]

Properties:
1. \( x(a, b) + y(a, b) + z(a, b) + w(a, b) \equiv 0 \!(\text{mod} \, 6) \)
2. \( x(a, -1) + R(a, -1) = 4a \)
3. \( z(a(a + 1), 2a + 1) \equiv x(a(a + 1), 2a + 1) + y(a(a + 1), 2a + 1) - R(a(a + 1), 2a + 1) = 24P^a \)
4. \( 3R(b + 1, b) - 3y(b + 1, b) - 36t_{a,b} \) is a nasty number.

Pattern 6:

Introduction of the linear transformations

\[ R = X + T \quad v = X + 5T \quad u = 2U \quad (13) \]

in (3) leads to \( U^2 = X^2 - 5T^2 \)

which is satisfied by

\[ X = r^2 + 5s^2 \]
\[ u = 2(r^2 - 5s^2) \]
\[ T = 2rs \]

Substituting the above values of \( X, u \) and \( T \) in (13), the corresponding non-zero distinct integral solutions of (3) are given by

\[ R = R(a, b) = r^2 + 5s^2 + 2rs \]
\[ v = v(a, b) = r^2 + 5s^2 + 10rs \]

Thus the corresponding solutions of (1) are found to be

\[ x = x(a, b) = 3r^2 - 5s^2 + 10rs \]
\[ y = y(a, b) = r^2 - 15s^2 - 10rs \]
\[ z = z(a, b) = 5r^2 - 15s^2 + 10rs \]
\[ w = w(a, b) = 3r^2 - 25s^2 - 10rs \]
\[ R = R(a, b) = r^2 + 5s^2 + 2rs \]

Properties:
1. \( x(1, s) - w(1, s) = 2(24_2a, s - 1) \)
2. \( x(r, s) + y(r, s) + z(r, s) + w(r, s) \equiv 0 \!(\text{mod} \, 12) \)
3. \( x(r, r(r + 1)) + R(r, r(r + 1)) = t_{1, 2r} + 6P^r \)
4. \(z(r,s) - R(r,s) \equiv 0 \pmod{4}\)

5. Each of the following represents a nasty number:
   - \(-y(r,r) - z(r,r)\)
   - \(3\{x(r,s) + w(r,s) - y(r,s) - z(r,s)\}\)

**REMARKABLE OBSERVATIONS**

I: \(\left[ \frac{2P_x^{\frac{1}{3}}}{t_{4,x-1}} \right]^4 - \frac{36P_y^{\frac{1}{3}}}{S_{y-2} - 1} \equiv 0 \pmod{5}\)

II: \(\left[ 5 \left( \frac{4P_x^{\frac{1}{3}}}{P_{z-3}^{\frac{1}{3}}} \right)^2 - 5 \left( \frac{6P_{z-1}^{\frac{1}{3}}}{t_{6,w+1}} \right)^2 \right] \left( \frac{t_{4,w-1}}{gn_w} \right)^2 + \left[ \frac{3P_y}{t_{3,y}} \right]^4\) is a bi-quadratic integer.

III: \(30 \left( \frac{4P_x^{\frac{1}{3}}}{C_t_{4,x-1}} \right)^4 - 30 \left( \frac{P_{z-1}^{\frac{1}{3}}}{t_{3,y-1}} \right)^4 - \left( \frac{P_{z-1}^{\frac{1}{3}}}{t_{3,z}} \right)^4 + 150 \left( \frac{CP_{w-1}}{t_{4,w}} \right)^2 \left( \frac{6P_{z-1}^{\frac{1}{3}}}{t_{3,2w-2}} \right)^2\) is a nasty number.

IV: If the non-zero integer quintuple \((x_0, y_0, z_0, w_0, R_0)\) is any solution of (1) then the quintuple \((x_n, y_n, z_n, w_n, R_n)\)

where

\[
x_n = u_0 + \tilde{y}_{n-1}v_0 + 5\tilde{x}_{n-1}R_0
\]
\[
y_n = u_0 - \tilde{y}_{n-1}v_0 - 5\tilde{x}_{n-1}R_0
\]
\[
z_n = 2u_0 + \tilde{y}_{n-1}v_0 + 5\tilde{x}_{n-1}R_0
\]
\[
w_n = 2u_0 - \tilde{y}_{n-1}v_0 - 5\tilde{x}_{n-1}R_0
\]
\[
R_n = \tilde{y}_{n-1}R_0 + \tilde{x}_{n-1}v_0
\]

also satisfies (1).In the above , \(u_0, v_0, R_0\) are the initial solutions of (3) and \((\tilde{x}_{n-1}, \tilde{y}_{n-1})\) is the solution of the pellian \(y^2 = 5x^2 + 1\)

**Note:**

In linear transformations (2), the variables \(z\) and \(w\) may also be represented by
\(z = 2uv + 1\) , \(w = 2uv - 1\)

Applying the procedure similar to that presented above in patterns 1 to 6, other choices of integer solutions of (1) are obtained.

**CONCLUSION**

To conclude, one may search for other patterns of solutions and their corresponding properties.

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