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ORIGINAL ARTICLE

On Second Order Duality in Nondifferentiable Multiobjective Programming

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ABSTRACT
A second order Mond-Weir type dual is formulated for a class of nondifferentiable multiobjective programming problem and duality results are proved involving second order generalized convexity assumptions.
Key words: Multiobjective programming; Nondifferentiable programming; Second order duality; Properly efficient solutions

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INTRODUCTION

The study of second order duality is significant due to the computational advantages over the first order duality, as it provides tighter bounds for the value of the objective function, when approximations are used [8,10,12]. Mangarsarian [10] considered a nonlinear program and discussed second order duality using certain inequalities. Mond [12] introduced the concept of second order convex function, which was named as bonvex function by Bector and Chandra [3]. Husain et al. [9] discussed second order duality results for non linear programs using Fritz John necessary conditions. Yang et al. [15] pointed out some inconsistencies in the statement and proof of the converse duality theorem given in [9].
Zhang and Mond [16] introduced second other (F, ρ)-convex functions and established duality theorems for second order Mangasarian, Mond-Weir and general Mond-Weir type vector duals. In [11], Mishra obtained weak and strong duality theorems for second order Mond-Weir type multiobjective dual involving generalized type I functions. Yang et al. [14] proposed a general Mond-Weir type dual for a class of nondifferentiable multiobjective program, in which each component of the objective function contains support function and derived a weak duality theorem under generalized convexity. Aghezza [1] formulated a second order mixed type dual and established various duality results with new class of second order generalized (F, ρ)-convex functions. Hachimi and Aghezzaf [7] established duality results for mixed type vector dual involving a new class of second order type I functions. Recently, Ahmad and Husain [2] introduced a class of second order (F, α, ρ, d)-convex functions and their generalizations, and established weak, strong and strict converse duality results for Mond-Weir type multiobjective dual.
The present paper is concerned with the following nondifferentiable multiobjective programming problem:

(P) Minimize $\left(f_1(x)+(x^tB_1x)^{\frac{1}{2}}, f_2(x)+(x^tB_2x)^{\frac{1}{2}}, \ldots, f_k(x)+(x^tB_kx)^{\frac{1}{2}} \right)$

Subject to $x \in S = \{x \in X : g(x) \leqq 0\}$,

where X is an open subset of R^n, $f_i : X \to R$, $i \in K$ and $g : X \to R^m$ are twice differentiable functions and B_i, $i \in K$ is an $n \times n$ positive semidefinite symmetric matrix.

In this paper, we formulate a second order Mond-Weir type dual for (P) and prove weak, strong, converse and strict converse duality theorems under second order generalized convexity assumptions. This work extends an earlier work of [2] and the references cited therein.

2. Notations and preliminaries

Throughout the paper, the following notations for vectors $x,y \in R^n$ will be used:

$x \geqq y$ if and only if $x_i \geqq y_i$, $i = 1,2, \ldots, n$;

$x \geq y$ if and only if $x \geqq y$ and $x \neq y$;

$x > y$ if and only if $x_i > y_i$, $i = 1,2, \ldots, n$.

Let $K = \{1,2,\ldots, k\}$ and for each fixed $r \in K$, $K_r = K - \{r\}$.

Consider the following vector minimum problem:

(VMP) Minimize $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$

subject to $x \in S$,

where $f : X \rightarrow R^k$ and $X \subseteq R^n$.

Definition 1. A point $\bar{x} \in S$ is said to be an efficient solution of (VMP), if there exits no other $x \in S$ such that

$$f_i(x) \leqq f_i(\bar{x}), i \in K_r$$

and

$$f_r(x) < f_r(\bar{x}), r \in K.$$

Definition 2. A point $\bar{x} \in S$ is said to be a weakly efficient solution of (VMP), there exits no other $x \in S$ with

$$f_i(x) < f_i(\bar{x}), i \in K_r$$

Definition 3 [5]. An efficient solution $\bar{x}$ of (VMP) is said to be properly efficient, if there exists a scalar $N > 0$ such that for each $r \in K$, $f_r(x) < f_r(\bar{x})$ and $x \in S$,

$$\frac{f_r(\bar{x}) - f_r(x)}{f_i(x) - f_i(\bar{x})} \leqq N$$

for at least one $i \in K_r$ such that $f_i(\bar{x}) < f_i(x)$.

Definition 4. A function $F : X \times X \times R^n \mapsto R$ is said to be sublinear in its third component, if for all x, $\bar{x} \in X$.

(i) $F(x, \bar{x}; \zeta_1 + \zeta_2) \leqq F(x, \bar{x}; \zeta_1) + F(x, \bar{x}; \zeta_2)$, for all $\zeta_1, \zeta_2 \in R^n$,

(ii) $F(x, \bar{x}; \beta\zeta_1) = \beta F(x, \bar{x}; \zeta_1)$ for all $\beta \in R$, $\beta \geqq 0$, and for all $\zeta_1 \in R^n$.

In the sequel, we require the following definitions [2].

Let F be sublinear, the function $f = (f_1, f_2, \ldots, f_k): X \mapsto R^k$ be differentiable at $\bar{x} \in X$ and let $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k$.

Definition 5. A twice differentiable function f_i over X is said to be second order (F, α, ρ_i, d)-convex at $\bar{x}$ on X, if for all $x \in X$, there exist a vector $p \in R^n$, a real valued function α: $X \times X \mapsto R_+ \setminus \{0\}$, a real valued function d (.,.): $X \times X \mapsto R$ and a real number ρ_i such that

$$f_i(x) - f_i(\bar{x}) + \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p \geqq F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) + \rho_i d^2(x, \bar{x}).$$

If the above inequality is strict, then f_i is said to be strictly second order (F, α, ρ_i, d)-convex at $\bar{x}$ on X.

A vector valued function $f = (f_1, f_2, \ldots, f_k)$: $X \to R^k$ is (strictly) second order (F, α, ρ_i, d)-convex at $\bar{x}$ on X, if each of its components f_i is (strictly) second order (F, α, ρ_i, d)-convex at $\bar{x}$ on X.

Definition 6. A twice differentiable function f_i over X is said to be second order (F, α, ρ_i, d)-pseudoconvex at $\bar{x}$ on X, if for all $x \in X$, there exist a vector $p \in R^n$, a real valued function α: $X \times X \mapsto R_+ \setminus \{0\}$, a real valued function d(.,.): $X \times X \mapsto R$ and a real number ρ_i such that

$$F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) \geqq -\rho_i d^2(x, \bar{x}) \Rightarrow f_i(x) \geqq f_i(\bar{x}) - \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p,$$

or equivalently,

$$f_i(x) < f_i(\bar{x}) - \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p \Rightarrow F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) < -\rho_i d^2(x, \bar{x}).$$

Definition 7. A twice differentiable function f_i over X is said to be strictly second order (F, α, ρ_i, d)-pseudoconvex at $\bar{x}$ on X, if for all $x \in X$, there exist a vector $p \in R^n$, a real valued function α: $X \times X \mapsto R_+ \setminus \{0\}$, a real valued function d(.,.): $X \times X \mapsto R$ and a real number ρ_i such that

$$F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) \geqq -\rho_i d^2(x, \bar{x}) \Rightarrow f_i(x) > f_i(\bar{x}) - \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p,$$

or equivalently,

$$f_i(x) \leqq f_i(\bar{x}) - \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p \Rightarrow F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) < -\rho_i d^2(x, \bar{x}).$$

A vector valued function $f = (f_1, f_2, \ldots, f_k)$: $X \to R^k$ is (strictly) second order (F, α, ρ_i, d)-psedoconvex at $\bar{x}$ on X, if each of its components f_i is (strictly) second order (F, α, ρ_i, d)-pseudoconvex at $\bar{x}$ on X.

Definition 8. A twice differentiable function f_i over X is said to be second order (F, α, ρ_i, d)-quasiconvex at $\bar{x}$ on X, if for all $x \in X$, there exist a vector $p \in R^n$, a real valued function α: $X \times X \mapsto R_+ \setminus \{0\}$, a real valued function d(.,.): $X \times X \mapsto R$ and a real number ρ_i such that

$$f_i(x) \leqq f_i(\bar{x}) - \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p \Rightarrow F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) \leqq -\rho_i d^2(x, \bar{x}),$$

or equivalently,

$$F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p\}) > -\rho_i d^2(x, \bar{x}) \Rightarrow f_i(x) > f_i(\bar{x}) - \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p.$$

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A vector valued function $f = (f_1, f_2,..., f_k): X \to R^k$ is second order (F, α, ρ_i, d)-quasiconvex at $\bar{x}$ on X, if each of its components f_i is second order (F, α, ρ_i, d)-quasiconvex at $\bar{x}$ on X.

An example of second order (F, α, ρ_i, d)-convex function is given by Ahmad and Husain [2] but they did not provide the examples of second order (F, α, ρ_i, d)-pseudoconvex functions and second order (F, α, ρ_i, d)-quasiconvex functions. The examples of such functions are given below.

Example 1. Define a function $f: X \to R$, where $X = \{x \in \mathrm{R} \mid x \geq 1\}$ by $f(x) = x^3 + 4x$ and $F: X \times X \times R \to R$ by $F(x, \bar{x}; a) = a\left(\frac{1}{x} + \bar{x}\right)$, distance function $d(x, \bar{x}) = x - \bar{x}, \rho = -4, 0 < p < \infty$, and $\alpha(x, \bar{x}) = x^2 + 1$. f is second order (F, α, ρ, d)-pseudoconvex function at $\bar{x} = 1$, but not second order (F, α, ρ, d)-convex. Also, when $\alpha(x, \bar{x}) = 1$, then f is not second order (F, ρ)-pseudoconvex.

Example 2. Define a function $f: X \to R$, where $X = \{x \in \mathrm{R} \mid x \geq 1\}$ by $f(x) = -x^3 + x^2$ and $F: X \times X \times R \to R$ by $F(x, \bar{x}; a) = a(x + \bar{x})$, distance function $d(x, \bar{x}) = x - \bar{x}, \rho = 2, 0 < p < \infty$, and $\alpha(x, \bar{x}) = x - \bar{x} + 1$. Then f is second order (F, α, ρ, d)-quasiconvex at $\bar{x} = 1$, but not second order (F, α, ρ, d)-convex. Also, when $\alpha(x, \bar{x}) = 1$, then f is not second order (F, ρ)-quasiconvex.

The following theorem [13] will be needed in the sequel:

Theorem 1. Let $\bar{x}$ be a properly efficient solution of (P) at which a constraint qualification is satisfied. Then there exist $\bar{\lambda} \in R^k$, $\bar{u} \in R^m$, $\bar{v}_i \in R^n$, $i \in K$ such that

$$\sum_{i \in K} \bar{\lambda}_i (\nabla f_i(\bar{x}) + B_i \bar{v}_i) + \nabla \bar{u}^t g(\bar{x}) = 0,$$

$$\bar{u}^t g(\bar{x}) = 0,$$

$$(\bar{x}^t B_i \bar{x})^{\frac{1}{2}} = \bar{x}^t B_i \bar{v}_i, i \in K,$$

$$\bar{v}_i^t B_i \bar{v}_i \leqq 1, \quad i \in K,$$

$$\bar{\lambda} > 0, \sum_{i \in K} \bar{\lambda}_i = 1, \ \bar{u} \geqq 0.$$

3. Second Order duality

In this section, we formulate the following Mond-Weir type dual associated to (P) and discuss duality results.

(MD) Maximize $\left(f_1(y) + y^t B_1 v_1 - \frac{1}{2} p^t \nabla^2 f_1(y) p, ..., f_k(y) + y^t B_k v_k - \frac{1}{2} p^t \nabla^2 f_k(y) p \right)$

Subject to $\sum_{i \in K} \bar{\lambda}_i (\nabla f_i(y) + \nabla^2 f_i(y) p + B_i v_i) + \nabla u^t g(y) + \nabla^2 u^t g(y) p = 0, \quad (1)$

$$u^t g(y) - \frac{1}{2} p^t \nabla^2 u^t g(y) p \geqq 0, \quad (2)$$

$$v_i^t B_i v_i \leqq 1, \ i \in K, \quad (3)$$

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$\lambda > 0,\ u \geqq 0.$ (4)

Remark 1. Let $B_i = 0, i \in K$. Then (MD) becomes the second order Mond-Weir type dual obtained by Zhang and Mond [16] with the addition of $\sum_{i\in K} \lambda_i = 1$. If, in addition, $p = 0$, then the dual (MD) reduces to the first order Mond-Weir type dual considerd in [6].

Let G and H denote the sets of all feasible solutions of (P) and (MD) respectively.

Theorem 2 (Weak Duality). Let $x \in G$ and $(y, u, v_1, v_2, ..., v_k, \lambda, p) \in H$. if

(i) $\sum_{i\in K} \lambda_i (f_i(.) + (.)^t B_i v_i)$ is second order (F, α, ρ, d)-pseudoconvex at y;

(ii) $u^t g(.)$ is second order (F, α_1, ρ_1, d)-quasiconvex at y; and

(iii) $\dfrac{\rho}{\alpha(x,y)} + \dfrac{\rho_1}{\alpha_1(x,y)} \geqq 0.$

Then the following cannot hold:

$$f_i(x) + (x^t B_i x)^{\frac{1}{2}} \leqq f_i(y) + y^t B_i v_i - \frac{1}{2} p^t \nabla^2 f_i(y) p,\ i \in K_r, \tag{5a}$$

and

$$f_r(x) + (x^t B_r x)^{\frac{1}{2}} < f_r(y) + y^t B_r v_r - \frac{1}{2} p^t \nabla^2 f_r(y) p,\ r \in K_r, \tag{5b}$$

Proof. For every $x \in G$ and $(y, u, v_1, v_2, ..., v_k, \lambda, p) \in H$,

$$u^t g(x) \leqq 0 \leqq u^t g(y) - \frac{1}{2} p^t \nabla^2 u^t g(y) p,$$

which by the virtue of hypothesis (ii) implies

$$F(x, y; \alpha_1(x, y)\ (\nabla^2 u^t g(y) + \nabla^2 u^t g(y)p)) \leqq -\rho_1 d^2(x,y).$$

As $\alpha_1(x, y) > 0$, it follows that

$$F(x, y; \nabla u^t g(y) + \nabla^2 u^t g(y)p) \leqq \frac{-\rho_1}{\alpha_1(x,y)} d^2(x,y).$$

The sublinearity of F along with (1) and (6), yields

$$0 = F(x, y; \sum_{i\in K} \lambda_i (\nabla f_i(y) + \nabla^2 f_i(y) p + B_i v_i) + \nabla u^t g(y) + \nabla^2 u^t g(y) p)$$

$$\leqq F(x, y; \sum_{i\in K} \lambda_i (\nabla f_i(y) + \nabla^2 f_i(y) p + B_i v_i) + F(x, y; \nabla u^t g(y) + \nabla^2 u^t g(y) p)$$

$$\leqq F(x,y;\sum_{i\in K}\lambda_i(\nabla f_i(y)+\nabla^2 f_i(y)p+B_i v_i))-\frac{\rho_1}{\alpha_1(x,y)}d^2(x,y),$$

or $$F(x,y;.\sum_{i\in K}\lambda_i(\nabla f_i(y)+\nabla^2 f_i(y)p+B_i v_i)) \geqq \frac{\rho_1}{\alpha_1(x,y)}d^2(x,y),$$

which by hypothesis (iii) gives

$$F(x,y;\sum_{i\in K}\lambda_i(\nabla f_i(y)+\nabla^2 f_i(y)p+B_i v_i)) \geqq -\frac{\rho}{\alpha\ (x,y)}d^2(x,y).$$

On using hypothesis (i) we, obtain

$$\sum_{i\in K}\lambda_i(f_i(x)+x^t B_i v_i) \geqq \sum_{i\in K}\lambda_i(f_i(y)+y^t B_i v_i-\frac{1}{2}p^t\nabla^2 f_i(y)p). \tag{7}$$

Now let x^* = $B_i^{1/2}x, i\in K$ and $v_i^* = B_i^{1/2}v_i, i\in K$. From Schwartz inequality and $v_i^{*t}v_i^* = v_i^t B_i v_i \leqq 1, i\in K,$

$$x^t B_i v_i = x^{*t}v_i^* \leqq \|x^*\|\|v_i^*\| \leqq \|x^*\| = (x^t B_i x)^{1/2}. \tag{8}$$

The equations (7) and (8) imply that

$$\sum_{i\in K}\lambda_i(f_i(x)+(x^t B_i x)^{\frac{1}{2}}) \geqq \sum_{i\in K}\lambda_i(f_i(y)+y^t B_i v_i-\frac{1}{2}p^t\nabla^2 f_i(y)p). \tag{9}$$

Suppose to the contrary that (5a) and (5b) hold, i.e.,

$$f_i(x)+(x^t B_i x)^{\frac{1}{2}} \leqq f_i(y)+y^t B_i v_i-\frac{1}{2}p^t\nabla^2 f_i(y)p,\ \ i\in K_r,$$

and

$$f_r(x)+(x^t B_r x)^{\frac{1}{2}} < f_r(y)+y^t B_r v_r-\frac{1}{2}p^t\nabla^2 f_r(y)p,\ \ r\in K.$$

Since λ > 0, therefore

$$\sum_{i\in K}\lambda_i(f_i(x)+(x^t B_i x)^{\frac{1}{2}}) < \sum_{i\in K}\lambda_i(f_i(y)+y^t B_i v_i-\frac{1}{2}p^t\nabla^2 f_i(y)p),$$

which is a contradiction to (9).

Theorem 3 (Strong Duality). Let $\bar{x}$ be a properly efficient solution of (P) at which a constraint qualification is satisfied. Then there exists $\bar{\lambda}$ ∈ R^k, $\bar{u}$ ∈ R^m, $\bar{v}_i$ ∈ R^n, i ∈ K and p ∈ R^n such that $(\bar{x}, \bar{u}, \bar{v}_1, \bar{v}_2, ..., \bar{v}_k, \bar{\lambda}, \bar{p}=0) \in H$ and the objective function values of (P) and (MD) are equal. Also, if the hypotheses of weak duality (Theorem 2) hold, then $(\bar{x}, \bar{u}, \bar{v}_1, \bar{v}_2, ..., \bar{v}_k, \bar{\lambda}, \bar{p}=0)$ is a properly efficient solution of (MD).

Proof. Since $\bar{x}$ is a properly efficient solution of (P) at which a constraint qualification is satisfied, by Theorem 1, there exist $\bar{\lambda} \in R^k$, $\bar{u} \in R^m$, $\bar{v}_i \in R^n$, $i \in K$ such that

$$\sum_{i\in K} \bar{\lambda}_i (\nabla f_i(\bar{x}) + B_i \bar{v}_i) + \nabla \bar{u}^t g(\bar{x}) = 0,$$

$$\bar{u}^t g(\bar{x}) = 0,$$

$$(\bar{x}^t B_i \bar{x})^{\frac{1}{2}} = \bar{x}^t B_i \bar{v}_i, \; i \in K,$$

$$\bar{u}^t B_i \bar{v}_i \leqq 1, i \in K,$$

$$\bar{\lambda} > 0, \; \bar{u} \geqq 0.$$

Thus $(\bar{x}, \bar{u}, \bar{v}_1, \bar{v}_2, ..., \bar{v}_k, \bar{\lambda}, \bar{p} = 0) \in H$ and the objective function values of (P) and (MD) are equal. Now we first show that $(\bar{x}, \bar{u}, \bar{v}_1, \bar{v}_2, ..., \bar{v}_k, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution of (MD). For this, assume that it is not efficient. Then there exists a feasible solution (y^*, u^*, v_1^*, v_2^*, ... , y_k^*, λ^*, p^*) such that

$$f_i(y^*) + x^{*t} B_i v_i^* - \frac{1}{2} p^{*t} \nabla^2 f_i(y^*) p^* \;\geqq\; f_i(\bar{x}) + \bar{x}^t B_i \bar{v}_i, \; i \in K_r,$$

and

$$f_r(y^*) + x^{*t} B_r v^*{}_r - \frac{1}{2} p^{*t} \nabla^2 f_r(y^*) p^* \;\geqq\; f_r(\bar{x}) + \bar{x}^t B_r \bar{v}_r, \; r \in K.$$

Using $(\bar{x}^t B_i \bar{x})^{\frac{1}{2}} = \bar{x}^t B_i \bar{v}_i, \; i \in K$ in the above inequalities, we get

$$f_i(\bar{x}) + (\bar{x}^t B_i \bar{x})^{\frac{1}{2}} \;\leqq\; f_i(y^*) + x^{*t} B_i v_i^* - \frac{1}{2} p^{*t} \nabla^2 f_i(y^*) p^*, \; i \in K_r,$$

and

$$f_r(\bar{x}) + (x^t B_r \bar{x})^{\frac{1}{2}} \;<\; f_r(y^*) + x^{*t} B_r v_r^* - \frac{1}{2} p^{*t} \nabla^2 f_r(y^*) p^*, \; r \in K,$$

Which contradicts weak duality (Theorem 2). Hence $(\bar{x}, \bar{u}, \bar{v}_1, \bar{v}_2, ..., \bar{v}_k, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution of (MD). Assume now that it is not properly efficient. Then there exists a feasible solution (y^*, u^*, v_1^*, v_2^*, ... , y_k^*, λ^*, p^*) and an $r \in K$ such that

$$f_r(y^*) + x^{*t} B_r v_r^* - \frac{1}{2} p^{*t} \nabla^2 f_r(y^*) p^* \;\geqq\; f_r(\bar{x}) + \bar{x}^t B_r \bar{v}_r$$

and

$$[f_r(y^*) + x^{*t} B_r v_r^* - \frac{1}{2} p^{*t} \nabla^2 f_r(y^*) p^*] - [f_r(\bar{x}) + \bar{x}^t B_r \bar{v}_r]$$

$$> M\left[[f_i(\bar{x})+\bar{x}^tB_i\bar{v}_i]-[f_i(y^*)+x^{*t}B_iv_i^*-\frac{1}{2}p^{*t}\nabla^2 f_i(y^*)p^*]\right],$$

for all $M > 0$ and all $i \in K_r$ satisfying

$$f_i(\bar{x})+\bar{x}^tB_i\bar{v}_i > f_i(y^*)+x^{*t}B_iv_i^*-\frac{1}{2}p^{*t}\nabla^2 f_i(y^*)p^*.$$

Again by $(\bar{x}^tB_i\bar{x})^{\frac{1}{2}}=\bar{x}^tB_i\bar{v}_i,\ \ i\in K$, we have

$$[f_r(y^*)+x^{*t}B_rv_r^*-\frac{1}{2}p^{*t}\nabla^2 f_r(y^*)p^*]-[f_r(\bar{x})+(\bar{x}^tB_r\bar{x})^{\frac{1}{2}}]$$

$$> M\left[[f_i(\bar{x})+(\bar{x}^tB_i\bar{x})^{\frac{1}{2}}]-[f_i(y^*)+x^{*t}B_iv_i^*-\frac{1}{2}p^{*t}\nabla^2 f_i(y^*)p^*]\right].$$

This means that $[f_r(y^*)+x^{*t}B_rv_r^*-\frac{1}{2}p^{*t}\nabla^2 f_r(y^*)p^*]-[f_r(\bar{x})+(\bar{x}^tB_r\bar{x})^{\frac{1}{2}}]$, $r \in K$ can be made arbitrary large, whereas $[f_i(\bar{x})+(\bar{x}^tB_i\bar{x})^{\frac{1}{2}}]-[f_i(y^*)+x^{*t}B_iv_i^*\ \ -\frac{1}{2}p^{*t}\nabla^2 f_i(y^*)p^*]$ is finite for all $i \in K_r$. Therefore,

$$f_i(\bar{x})+(\bar{x}^tB_i\bar{x})^{\frac{1}{2}}\ \leqq\ f_i(y^*)+x^{*t}B_iv_i^*-\frac{1}{2}p^{*t}\nabla^2 f_i(y^*)p^*,\ \ i\in K_r,$$

and

$$f_r(\bar{x})+(\bar{x}^tB_r\bar{x})^{\frac{1}{2}}\ <\ f_r(y^*)+x^{*t}B_rv_r^*-\frac{1}{2}p^{*t}\nabla^2 f_r(y^*)p^*,\ \ r\in K,$$

again a contradiction to weak duality (Theorem 2). Hence $(\bar{x},\,\bar{u},\,\bar{v}_1,\,\bar{v}_2,\dots,\bar{v}_k,\bar{\lambda},\ \bar{p}=0)$ is a properly efficient solution of (MD).

Theorem 4 (Converse Duality). Let $(\bar{y},\,\bar{u},\,\bar{v}_1,\,\bar{v}_2,\dots,\bar{v}_k,\bar{\lambda},\bar{p}=0)$ be a weakly efficient solution of (MD). Assume that

(i) either (a) then $n \times n$ Hessian matrix $\nabla^2(\bar{u}^tg(\bar{y}))$ is a positive definite and $\bar{p}^t\nabla\bar{u}^tg(\bar{y})\ \geqq\ 0$ or, (b) the $n \times n$ Hessian matrix $\nabla^2(\bar{u}^tg(\bar{y}))$ is a negative definite and $\bar{p}^t\nabla\bar{u}^tg(\bar{y})\ \leqq\ 0$.

(ii) the vectors $\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i,\ i\in K$ are linearly independent vectors; and

(iii) the vectors $\{[\nabla^2 f_i(\bar{y})]_j,[\nabla^2\bar{u}^tg(\bar{y})]_j\,;\,i\in K,\ j=1,2,\dots,n\}$ are linearly independent, where $[\nabla^2 f_i(\bar{y})]_j$ is the j-th row of $\nabla^2 f_i(\bar{y})$ and $[\nabla^2\bar{u}^tg(\bar{y})]_j$ is the j-th row of $\nabla^2\bar{u}^tg(\bar{y})$.

Then $\bar{y}$ is a properly efficient solution of (P) and the objective values of (P) and (MD) are equal.

Proof. Since $(\bar{y}, \bar{u}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k, \bar{\lambda}, \bar{p} = 0)$ is a weakly efficient solution of (MD), then by Firtz John conditions [4], there exist $\alpha \in R^k$, $\beta \in R^n$, $\gamma \in R$, $\xi \in R$, $\eta \in R^k$ and $\delta \in R^k$ such that

$$\sum_{i \in K} \alpha_i [\nabla f_i(\bar{y}) + B_i \bar{v}_i - \frac{1}{2} \bar{p}^t \nabla(\nabla^2 f_i(\bar{y})\bar{p})] - \beta^t [\nabla^2 \sum_{i \in K} \bar{\lambda}_i f_i(\bar{y})$$

$$+ \nabla(\nabla^2 \sum_{i \in K} \bar{\lambda}_i f_i(\bar{y})\bar{p}) + \nabla^2 \bar{u}^t g(\bar{y}) + \nabla(\nabla^2 \bar{u}^t g(\bar{y})\bar{p})]$$

$$+ \gamma[\nabla \bar{u}^t g(\bar{y}) - \frac{1}{2} \bar{p}^t \nabla(\nabla^2 \bar{u}^t g(\bar{y})\bar{p})] = 0, \quad (10)$$

$$-\beta^t [\nabla g(\bar{y}) + \nabla^2 g(\bar{y})\bar{p}] + \gamma[g(\bar{y}) - \frac{1}{2} \bar{p}^t \nabla^2 g(\bar{y})\bar{p}] + \delta = 0, \quad (11)$$

$$\alpha_i \bar{y}^t B_i - \bar{\lambda}_i \beta^t B_i - 2\xi B_i \bar{v}_i = 0, \ \ i \in K, \quad (12)$$

$$-\beta^t \left(\nabla f_i(\bar{y}) + \nabla^2 f_i(\bar{y})\bar{p} + B_i \bar{v}_i\right) + \eta_i = 0, \ i \in K, \quad (13)$$

$$\sum_{i \in K} \left(\alpha_i \bar{p} + \bar{\lambda}_i \beta\right)^t \nabla^2 f_i(\bar{y}) + (\gamma \bar{p} + \beta)^t \nabla^2 \bar{u}^t g(\bar{y}) = 0, \quad (14)$$

$$\gamma\left[\bar{u}^t g(\bar{y}) - \frac{1}{2} \bar{p}^t \nabla^2 \bar{u}^t g(\bar{y})\bar{p}\right] = 0, \quad (15)$$

$$\xi\left[\bar{v}_i^t B_i \bar{v}_i - 1\right] = 0, \ i \in K, \quad (16)$$

$$\eta^t \bar{\lambda} = 0, \quad (17)$$

$$\delta^t \bar{u} = 0, \quad (18)$$

$$(\alpha, \gamma, \xi, \eta, \delta) \geqq 0, \quad (19)$$

$$(\alpha, \beta, \gamma, \xi, \eta, \delta) \neq 0, \quad (20)$$

Equation (14) along with hypothesis (iii) implies

$$\alpha_i \bar{p} + \bar{\lambda}_i \beta = 0, \ i \in K \quad (21)$$

and

$$\gamma \bar{p} + \beta = 0. \quad (22)$$

Using (1), (21) and (22) in (10), we have

$$\sum_{i\in K}(\alpha_i-\lambda_i\gamma)\left[\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i\right]-\left[\nabla\left\{\nabla^2\bar{\lambda}^t f(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\right\}\bar{p}\right]\beta$$

$$-\frac{1}{2}\sum_{i\in K}\nabla\left\{(\alpha_i\bar{p})^t\nabla^2 f_i(\bar{y})\bar{p}\right\}-\frac{1}{2}\nabla\left\{(\gamma\bar{p})^t\nabla^2\bar{u}^t g(\bar{y})\bar{p}\right\}=0. \qquad (23)$$

Let $\gamma = 0$. Then equation (22) yields $\beta = 0$ and equation (21) gives

$$\alpha_i\bar{p}=0,\ i\in K.$$

Thus equation (23) reduces to

$$\sum_{i\in K}\alpha_i\left[\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i\right]=0,$$

which by hypothesis (ii) gives

$$\alpha_i=0,\ i\in K.$$

Also, equations (11) and (13) yield

$$\delta=0 \text{ and } \eta_i=0,\ i\in K.$$

Now, equation (12) along with (16) imply $\xi = 0$. Hence $(\alpha, \beta, \gamma, \xi, \eta, \delta) = 0$. A contradiction to (20). Therefore $\gamma > 0$.

In the similar way, one can obtain $\alpha_i > 0$, $i \in K$, by exhibiting a contradiction.

On multiplying (11) by $\bar{u}^t$ and then using (15) and (18), we get

$$\beta^t\left(\nabla\bar{u}^t g(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\bar{p}\right)=0.$$

which with (22) yields

$$\gamma\bar{p}^t\left(\nabla\bar{u}^t g(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\bar{p}\right)=0$$

or

$$\bar{p}^t\nabla\left(\bar{u}^t g(\bar{y})\right)=-\bar{p}^t\nabla^2\left(\bar{u}^t g(\bar{y})\right)\bar{p}.$$

This contradicts hypothesis (i) for $\bar{p}\neq 0$. Therefore $\bar{p}=0$ and then by (21), we have $\beta = 0$. On using $\bar{p}=0$ and $\beta=0$, equation (11) reduces to

$$g(\bar{y})=-\frac{\delta}{\gamma}.$$

Since $\delta \geqq 0$ and $\gamma > 0$, we have

$$g(\bar{y})\leqq 0.$$

Hence, $\bar{y}$ is feasible for (P).

Also, β = 0, $\alpha_i > 0,\ i \in K$ and (12) give

$$B_i\bar{y} = \frac{2\xi}{\alpha_i} B_i\bar{v}_i,\ i \in K, \tag{24}$$

and hence

$$\bar{y}^t\ B_i\bar{v}_i = \left(\bar{y}^t B_i\bar{y}\right)^{\frac{1}{2}} \left(\bar{v}_i^t\ B_i\bar{v}_i\right)^{\frac{1}{2}},\ i \in K. \tag{25}$$

If $\xi > 0,$ then (16) implies $\bar{v}_i^t B_i \bar{v}_i = 1$ and so (25) gives

$$\bar{y}^t B_i v_i = \left(\bar{y}^t B_i \bar{y}\right)^{\frac{1}{2}},\ i \in K.$$

Hence, in either case, we have

$$\bar{y}^t B_i\bar{v}_i = \left(\bar{y}^t B_i\bar{y}\right)^{\frac{1}{2}},\ i\in K. \tag{26}$$

Therefore by (26) and $\bar{p} = 0,$ we get

$$f_i(\bar{y}) + \left(\bar{y}B_i\bar{y}\right)^{\frac{1}{2}} = f_i(\bar{y}) + \bar{y}^t B_i\bar{v}_i - \frac{1}{2}\bar{p}^t\nabla^2 f_i(\bar{y})\bar{p}.$$

Thus by weak duality (Theorem 2), $\bar{y}$ is a properly efficient solution of (P).

Theorem 5 (Strict Converse Duality). Let $\bar{x} \in$ G and $\left(\bar{y},\bar{u},\bar{v}_1,\bar{v}_2,....,\bar{v}_k,\bar{\lambda},\bar{p}\right) \in H$ such that

$$\sum_{i\in K}\bar{\lambda}_i\ \left[f_i(x) + \bar{x}^t B_i\bar{v}_i\right] \leqq \sum_{i\in K}\bar{\lambda}_i\left[f_i(\bar{y}) + \bar{y}^t B_i\bar{v}_i - \frac{1}{2}\bar{p}^t\nabla^2 f_i(\bar{y})\bar{p}\right]. \tag{27}$$

If $\sum_{i\in K}\bar{\lambda}_i\ \left[f_i(.) + (.)^t B_i\bar{v}_i\right]$ is second order strictly $(F,\alpha,\rho,d)-$ pseudoconvex at $\bar{y},$ and $\bar{u}^t g(.)$ is second order $(F,\alpha,\rho,d)-$ quasiconvex at $\bar{y}$ with $\dfrac{\rho}{\alpha(\bar{x},\bar{y})} + \dfrac{\rho_1}{\alpha_1(\bar{x},\bar{y})} \geqq 0.$

Then $\bar{y} = \bar{x}$.

Proof. We assume that $\bar{y} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ and $\left(\bar{y},\bar{u},\bar{v}_1,\bar{v}_2,....,\bar{v}_{k},\bar{\lambda},\bar{\rho}\right)$ are feasible for (P) and (MD), we have

$$\bar{u}^t g(\bar{x}) \leqq 0 \leqq \bar{u}^t\ g\ (\bar{y}) - \frac{1}{2}\bar{p}^t\nabla^2\bar{u}^t g(\bar{y})\bar{p}.$$

Using second order $(F, \alpha_1, \rho_1\ d)-$ quasiconvexity of $\bar{u}^t g(.)$ at $\bar{y}$, we get

$$F\left(\bar{x},\bar{y};\ \alpha_1(\bar{x},\ \bar{y})\left(\nabla\bar{u}^t g(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\bar{p}\ \right)\leqq -\ \rho_1 d^2(\bar{x},\ \bar{y})\right)$$

which by $\alpha_1\ (\bar{x},\ \bar{y}) > 0\ yields$

$$F\left(\bar{x},\bar{y};\nabla\bar{u}^t(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\bar{p}\right)\leqq -\frac{\rho_1}{\alpha_1(\bar{x},\bar{y})}d^2(\bar{x},\ \bar{y}). \qquad (28)$$

The sublinearity of *F* and (1) imply

$$0 = F(\bar{x},\bar{y};\sum_{i\in K}\bar{\lambda}_i(\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i)+\nabla\bar{u}^t g(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\bar{p})$$

$$\leqq F(\bar{x},\bar{y};\sum_{i\in K}\bar{\lambda}_i(\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i)+F(\bar{x},\bar{y};\nabla\bar{u}^t g(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})\bar{p})$$

$$\leqq F(\bar{x},\bar{y};\sum_{i\in K}\bar{\lambda}_i(\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i))-\frac{\rho_1}{\alpha_1(\bar{x},\bar{y})}d^2(\bar{x},\bar{y}),$$

Since $\frac{\rho}{\alpha(\bar{x},\bar{y})}+\frac{\rho_1}{\alpha_1(\bar{x},\bar{y})} \geqq 0$, we obtain

$$F(\bar{x},\bar{y};.\sum_{i\in K}\bar{\lambda}_i(\nabla f_i(\bar{y})+\nabla^2 f_i(\bar{y})\bar{p}+B_i\bar{v}_i)) \geqq -\frac{\rho}{\alpha(\bar{x},\bar{y})}d^2(\bar{x},\bar{y}).$$

The second order strict (*F*, *α*, *ρ*, *d*)-pseudoconvexity of $\sum_{i\in K}\bar{\lambda}_i[f_i(.)+(.)^t B_i\bar{v}_i]$ at $\bar{y}$ gives

$$\sum_{i\in K}\bar{\lambda}_i[f_i(\bar{x})+\bar{x}^t B_i\bar{v}_i] > \sum_{i\in K}\bar{\lambda}_i(f_i(\bar{y})+\bar{y}^t B_i\bar{v}_i-\frac{1}{2}\bar{p}^t\nabla^2 f_i(\bar{y})\bar{p}),$$

a contradiction to (27). Hence $\bar{y}=\bar{x}$.

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