# Behavior of Degenerate one-particle state under Arbitrary Lorentz Transformation 

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#### Abstract

Application of time-reversal operator on one particle state produces time-reversed degenerate state with same energy. In this paper we will discuss effect of homogeneous Lorentz transformation on degenerate massless one-particle state. For a physical system the time-reversal operation is represented by an operator in the space of physical states. This operator changes a state into a different degenerate state of the same system. For a unitary operator representing an arbitrary Lorentz transformation of this state, expression for the standard Lorentz transformation has been obtained. After calculating the little-group element, a simple expression for the phase factor has been derived. Calculation has been done making use of canonical homomorphism between Lorentz group and group SL (2.C). Choosing two specific axes of rotation, Lorentz transformation action has been studied for degenerate massless particle state.


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## INTRODUCTION

Discrete transformations like time-reversal and space-inversion play an important role in quantum field theory. Though we observe different views regarding the unitarity and linearity of the operators corresponding to these transformations ,one thing is for sure that they are integral part of the field theory and there is no doubt about their existence. As there are no negative energy states observed, we assume time-reversal operator antilinear and antiunitary. The transformation properties of free particle states under the unitary representation of poincare' group is essentially included in the corresponding little group. Wigner's little group [1] is the subgroup of homogeneous Lorentz group that leaves the energy-momentum four-vector for a particle invariant. For a massless particle, the little group is Euclidian group E (2) which consists of rotations and spatial translations. Since group E (2) is noncompact, a finite dimensional unitary representation of it is not possible. To get a finite dimensional unitary representation, all non-compact generators of the group E (2) are set equal to zero and we are left with the only generator which gives the helicity $\lambda$ of the massless particle. Helicity gives the component of angular momentum in the direction of motion . To study the Lorentz transformation properties of general massless particle states, expression for phase factor is required.

## LITTLE GROUP AND PHASE FACTOR ANALYSIS

The massless one-particle state $\mid \bar{k}^{\mu}, \lambda$ is defined as an eigenvector of $K^{\mu}$ with eigenvalue $\bar{k}^{\mu}=$ ( $k, 0,0, k$ ), and an eigenvector of three-component of angular-momentum three-vector with eigenvalue $\lambda$. In general, $K^{\mu}$ is four-momentum, $\bar{k}^{\mu}$ is standard four-momentum and $\lambda$ is the helicity [2]. Application of time-reversal operator on state $\mid \bar{k}^{\mu}, \lambda$ leads to another state of eigenvalue ( $k, 0,0, \mathrm{k}$ ) with helicity unchanged. This is a new state but with same energy and therefore must be degenerate with eigenstate $\mid \bar{k}^{\mu}, \lambda$. For a massless particle the momentum four-vector eigen-states transform according to formula [2] as

$$
\begin{equation*}
U(\Lambda)\left|k, \lambda=e^{i \lambda \varphi(\Lambda, k)}\right| \Lambda k, \lambda \tag{1}
\end{equation*}
$$

In this formula, $\Lambda$ denotes arbitrary Lorentz transformation, and $U(\Lambda)$ denotes unitary operator in the unitary representation of poincare' group. Helicity $\lambda$ takes integer and half-integer values only. In fact there are topological considerations that restrict the allowed values of $\lambda$ to integers and half-integers for massless particle. We observe that, under a transformation the phase $\varphi(\Lambda, k)$ plays an important role. We will make use of the canonical homomorphism between the group SL (2.C) and the Lorentz group to get an expression for the phase factor.
Any two dimensional Hermitian matrix $H$ can be used to define a real four-vector $k^{\mu}$ as[2]

$$
\begin{equation*}
\mathrm{H}=k^{\mu} \sigma_{\mu} \tag{2}
\end{equation*}
$$

Where $\sigma_{i}, \mathrm{i}=1,2,3$ are standard Pauli matrices with $\sigma_{o}$ is equal to identity matrix.
Let us define a real linear transformation of $k^{\mu}$, that leaves the covariant square of the four- vector $k^{\mu}$ invariant, as

$$
\begin{equation*}
\mathrm{H}^{\prime}=V \mathrm{H}^{\dagger} \tag{3}
\end{equation*}
$$

Where $V$ is the element of the SL(2.C) group and defines a homogeneous Lorentz transformation $\quad \Lambda(V)$. We now take case of massless particle and choose a standard four-vector [2] as $(1,0,0,1)$. For standard four-vector ( $1,0,0,1$ ) , an explicit matrix representation of Eq. (2) is

$$
\widetilde{\mathrm{H}}=\left(\begin{array}{ll}
0 & 0  \tag{4}\\
0 & 2
\end{array}\right)
$$

Now we define a subgroup of SL (2.C), an element of which leaves $\widetilde{H}$ invriant. An element of this subgroup can be obtained, by keeping the hermiticity of $H$ preserved and solving Eq. (3), as

$$
\tilde{V}=\left(\begin{array}{cc}
e^{i^{\varphi} / 2} & 0  \tag{5}\\
z & e^{-i^{\varphi} / 2}
\end{array}\right)
$$

$\varphi$ ranges from 0 to $2 \pi, z$ is an arbitrary complex number. The matrix $\tilde{V}$ produces a Lorentz transformation. In fact, the phase factor in the matrix produces spatial rotations about the three-axis and complex parameter $z$ is just translation part. Hence, subgroup to which $\tilde{V}$ belongs is isomorphic to Euclidian group E (2) .
Now Wigner's little-group element $W(\Lambda, k)$, where $\boldsymbol{\Lambda}$ is an arbitrary Lorentz transformation, leaves standard four-vector invariant[2]. In SL (2.C) the corresponding little-group element can be written as [2,3]

$$
\begin{equation*}
G(\Lambda, k)=V_{\Lambda k}^{-1} \Lambda V_{k} \tag{6}
\end{equation*}
$$

$\Lambda$ is an arbitrary Lorentz transformation in SL (2.C).
Structure of little-group element includes translations and rotations in two dimensions. Since a massless particle should not have any continuous degree of freedom like suppose $\boldsymbol{\omega}$, an angle representing a rotation around three-axis, the structure of little group element reduces to a rotation. To calculate the little-group element, we need to fix a suitable convention for standard Lorentz transformation $V_{k}$ [2].
$V_{k}$ can be chosen to have the following form:

$$
\begin{equation*}
V_{k=} R(\tilde{k}) B(u) \tag{7}
\end{equation*}
$$

where $B(u)$ is a boost along the negative three-direction, and $R(\tilde{k})$ is a rotation which carries the negative three-axis into the direction of unit vector $\tilde{\mathbf{k}}=\frac{\mathbf{k}}{|\mathbf{k}|}$.
Therefore,

$$
V_{k}=\frac{1}{\sqrt{2 k^{0}\left(1-\tilde{k}_{3}\right)}}\left(\begin{array}{cc}
(1 & \left.\tilde{k}_{3}\right)  \tag{8}\\
\tilde{k}_{+} & k^{0} \tilde{k}_{-} \\
k^{0}\left(\begin{array}{ll}
1 & \tilde{k}_{3}
\end{array}\right)
\end{array}\right)
$$

where $\tilde{k}_{ \pm}=\tilde{k}_{1} \pm i \tilde{k}_{2}$.
The phase of Lorentz transformation defined by $\tilde{V}$ is of real importance in transformation properties of states. $G(\Lambda, k)_{m, n}$, for $m=n=2$, can be used to calculate the phase factor for unitary operator representing Lorentz transformation in the unitary representation of the poincare' group.
An arbitrary Lorentz transformation in SL (2.C) can be of the form[2]

$$
\Lambda=\left(\begin{array}{ll}
\alpha & 0  \tag{9}\\
0 & \beta
\end{array}\right)
$$

$\Lambda$ Is complex unimodular matrix with condition $\alpha \beta=1$.
Using Eqs. (2), (3), (6), and (8) and making a straightforward calculation we get

$$
\begin{equation*}
\tilde{V}_{22}=\frac{-b \beta\left(1-\tilde{k}_{3}\right)-c \alpha \tilde{k}_{-}}{a \sqrt{b\left(1-\tilde{k}_{3}\right)}} \tag{10}
\end{equation*}
$$

Where

$$
\begin{align*}
& a \quad b=|\alpha|^{2}\left(1+\tilde{k}_{3}\right)  \tag{11}\\
& b=|\beta|^{2}\left(1 \quad \tilde{k}_{3}\right)  \tag{12}\\
& c=\alpha \beta \tilde{k}_{+}
\end{align*}
$$

Now expression for general phase factor can be obtained, using Eq. (10), as

$$
\begin{equation*}
e^{i \varphi}=\frac{\alpha\left(1-\tilde{k}_{3}\right)}{\beta\left(1-\tilde{k}_{3}\right)} \tag{14}
\end{equation*}
$$

It is evident from Eq. (1), that the phase factor is associated with corresponding unitary operator. So we relate $\alpha$ and $\beta$ with the elements of a general SU (2) member and insist on $\beta=\alpha$ [4]. However, when we consider only rotation, a generalization can be done regarding the values of $\alpha$ and $\beta$ in Eq. (9). We simply associate $\Lambda$ with $2 \times 2$ unitary unimodular matrix for real angle $\theta$ around the three-axis and get

$$
\begin{align*}
& e^{i \varphi}=\alpha / \alpha=e^{i \theta}  \tag{15}\\
& \varphi=\theta
\end{align*}
$$

Therefore, in the case of time-reversed states the phase $\varphi$ is equal to an arbitrary rotation angle observed around the three-axis. Now it is known that the structure of little group element, Eq. (6), has already been fixed for a rotation around the three-axis and standard Lorentz transformation takes the three-axis to the direction of unit vector $\tilde{\mathbf{k}}$. Therefore, for any arbitrary Lorentz transformation, like rotation, the rotation around the three-axis by an angle will be equal to the rotation around the direction of unit vector $\tilde{\mathbf{k}}$ by same amount. Here we want to mention the paper [5]. In this paper authors have taken case of massless particle under an arbitrary Lorentz transformation. They show that if the Lorentz transformation is a pure rotation, representation of generic little group must not contain the terms related to generators of translations in 12-plane. They have shown that the phase, like $\varphi$ in our case, is equal to an arbitrary rotation angle $\bar{\omega}$, like $\theta$, around three-axis, a similar conclusion which we have obtained. We now make a logical move to calculate the phase factor for transformation like boost in the direction of $\tilde{\mathbf{k}}$. For massless one-particle state, the non-compact generators are set equal to zero. This reduces the structure of little group to a rotational group and also introduces the Lorentz invariant quantity $\lambda$. Now that we have the particle with Lorentz invariant helicity; to calculate the phase factor we require only the value of $\varphi$ to be determined. Since, little group structure has already been reduced to rotation, and we are dealing with boost, obviously $\varphi=0$.

## CONCLUSION

Normal state of a physical system and corresponding time-reversed state appear different on a physical ground of observation. However, we observe that under the action of Lorentz transformation, these states, for massless particle, are simply indistinguishable from each other. This is explicitly exhibited by the equivalence of the expression for the phase angle in our case and in ref. [3].

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