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A Study on Fractional Differintegrations of Generalized Hypergeometric Functions

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ABSTRACT

In the present paper, the author has defined the fractional differintegrations of Generalized Wright Function in association with different functions of one variable. Corollary and some examples are also given.

Keywords: Goursat's theorem, Fractional Differintegrations

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INTRODUCTION

Definitions of the fractional derivatives and integral of the function of single variable:

(i) Goursat's theorem (Cauchy's theorem) for the function of single variable is:

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta \qquad (n\in N\cup\{0\}, z\in D) \tag{1.1}$$

Where $f(z)$ is analytic in a domain D, which is surrounded with a piecewise smooth closed Jorden curve γ, in the ζ-plane.

(ii) (Derivative). If $f(z)$ is an analytic (regular) function and it has no branch point inside $C(=\{\underset{-}{C},\underset{+}{C}\})$ and on C, and

$$_{\underset{-}{C}}f_{\nu}={}_{\underset{-}{C}}f_{\nu}(z)=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \tag{1.2}$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty}^{(0+)}\eta^{-(\nu+1)}f(z+\eta)d\eta, \quad (\zeta-z=\eta) \tag{1.3}$$

$$(\zeta\neq z, -\pi\leq\arg(\zeta-z)\leq\pi, \nu\notin Z^{-})$$

$$_{\underset{+}{C}}f_{\nu}={}_{\underset{+}{C}}f_{\nu}(z)=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \tag{1.4}$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty}^{(0+)}\eta^{-(\nu+1)}f(z+\eta)d\eta, \quad (\zeta-z=\eta) \tag{1.5}$$

$$(\zeta\neq z, -\pi\leq\arg(\zeta-z)\leq\pi, \nu\notin Z^{-})$$

$$f_{-n} = {}_C f_{-n} = \lim_{\nu \to -n} {}_C f_\nu \quad (n \in Z^+, C = \{\underset{-}{C}, \underset{+}{C}\}), \tag{1.6}$$

Where $\underset{-}{C}$ and $\underset{+}{C}$ are integral curves as shown in Fig. 1 and Fig. 2 (that is $\underset{-}{C}$ is a curve along the cut joining two points z and $-\infty + i\lim(z)$, and $\underset{+}{C}$ is a curve along the cut joining two points z and $\infty + i\lim(z)$, then $f_\nu = {}_C f_\nu(z) = \{{}_{\underset{-}{C}} f_\nu(z), {}_{\underset{+}{C}} f_\nu(z)\}(\nu > 0)$ is the fractional derivative of order ν of the function $f(z)$, if f_ν exists.

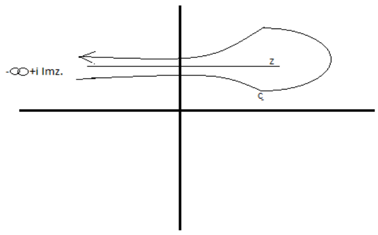
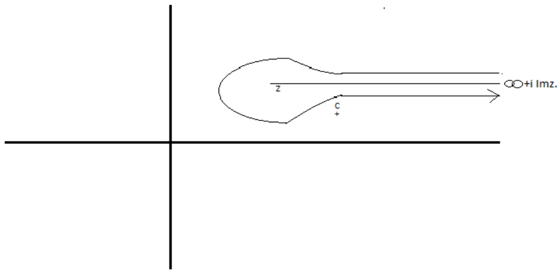


Fig.1



Fug.2

Definition 2 (Integral). $f_\nu(\nu < 0)$ is the fractional integral of order $|\nu|$. That is, the derivative of fractional order $-\nu(\nu > 0)$ is the fractional integral of order $\nu(\nu \in R)$, if f_ν exists.

Formal unification of derivative and integral of the function of single variable:

If $f(z)$ is the analytic function and it has no branch point inside C and on $C(C = \{\underset{-}{C}, \underset{+}{C}\})$, and

$$f_\nu = {}_C f_\nu(z) = \{{}_{\underset{-}{C}} f_\nu(z), {}_{\underset{+}{C}} f_\nu(z)\} \tag{1.7}$$

Then

$$f_\nu \text{ is } \begin{cases} \text{derivative } for\, \nu > 0 \\ original \;\; for\, \nu = 0 \\ \text{integral } \;\; for\, \nu < 0 \end{cases} \tag{1.8}$$

For $\nu \in R$, and

$$f_\nu \text{ is } \begin{cases} \text{derivative } for\, \text{Re}(\nu) > 0 \\ original \;\; for\, \nu = 0 \\ \text{integral } \;\; for\, \text{Re}(\nu) < 0 \end{cases} \tag{1.9}$$

For $\nu \in C$, if f_ν exists.

And in case of $\text{Re}(\nu) = 0, f_\nu$ is only formal differintegration regardless of $\text{Im}(\nu) \geq 0$ or $\text{Im}(\nu) \leq 0$. That is , we have no derivative and integral for $\nu =$pure imaginary.

Following results will be used:

(i) From ([1];p.16, eq.(1))

$$\left(e^{-az}\right)_\nu = e^{-i\pi\nu} a^\nu e^{-az} \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.10}$$

(ii) From([1];p.18, eq.(6))

$$\left(e^{az}\right)_\nu = a^\nu e^{-az} \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.11}$$

(iii) From ([1];p.19, eq.(11))

$$\left(a^z\right)_\nu = (\log a)^\nu a^z \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.12}$$

(iv) From ([1];p.20, eq.(1))

$$\left(\cosh az\right)_{\nu} = (-ia)^{\nu}\cosh(az + i\frac{\pi}{2}\nu) \quad \text{for } a \neq 0 (z,\nu \in C) \tag{1.13}$$

(v) From ([1];p.20, eq.(2))

$$\left(\sinh az\right)_{\nu} = (-ia)^{\nu}\sinh(az + i\frac{\pi}{2}\nu) \quad \text{for } a \neq 0 (z,\nu \in C) \tag{1.14}$$

(vi) From ([1];p.21, eq.(1))

$$\left(\cos az\right)_{\nu} = (a)^{\nu}\cos(az + \frac{\pi}{2}\nu) \text{ for } a \neq 0 (z,\nu \in C) \tag{1.15}$$

(vii) From ([1];p.22, eq.(2))

$$\left(\sin az\right)_{\nu} = (a)^{\nu}\sin(az + \frac{\pi}{2}\nu) \text{ for } a \neq 0 (z,\nu \in C) \tag{1.16}$$

(viii) From ([1];p.32, eq.(1))

$$(\log az)_{\nu} = -e^{-i\pi\nu}\Gamma(\nu)\, z^{-\nu} \text{ for } a \neq 0 (z,\nu \in C) \tag{1.17}$$

Generalized Wright Function ${}_2R_1(a,b;c,\omega;\mu;z)$ defined by Dotesnko (2,3)as:

$${}_2R_1(a,b;c,\omega;\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k\frac{\omega}{\mu})}{\Gamma(c+k\frac{\omega}{\mu})}\frac{z^k}{k!} \tag{1.18}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,{}_2\psi_1\left[z\left|\begin{matrix}(a,1),(b,\frac{\omega}{\mu})\\(c,\frac{\omega}{\mu})\end{matrix}\right.\right] \tag{1.19}$$

Wright function can be expressed by the Mellin-Barnes integral by Kilbas et.el. ([4], p.123)in the form:

$${}_2R_1(a,b;c,\omega;\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega}{\mu}s)}{\Gamma(c-\frac{\omega}{\mu}s)}(-z)^{-s}\,ds \quad (1.20))$$

Where the contour of integration $L = L_{-\infty}$ separates all poles of $\Gamma(-s)$ to the left and all the poles of $\Gamma(a-s)$ and $\Gamma(b-\frac{\omega}{\mu}s)$ to the right.

Main Results

Theorem 1.

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu} = e^{-i\pi\nu}(kz^{\nu})^{-1}\,{}_2F_1(a,b,;c;\omega,\mu;-e^{-kz}) \text{ for } k \neq 0 (z,\nu \in C)$$

Proof: In case of $|\arg k| < \frac{\pi}{2}$

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu} = \underset{+}{C}\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu}$$

$$= \frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{{}_2F_1(a,b;c;\omega,\mu;e^{-k\zeta})}{(\zeta-z)^{\nu+1}}\,d\zeta$$

$$= \frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{\left\{\frac{1}{2\pi i}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega}{\mu}s)}{\Gamma(c-\frac{\omega}{\mu}s)}(e^{-k\zeta})^s\,ds\right\}}{(\zeta-z)^{\nu+1}}\,d\zeta$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds\left\{\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{e^{-ks\zeta}}{(\zeta-z)^{\nu+1}}d\zeta\right\}$$

Where

$$\theta(s)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega}{\mu}s)}{\Gamma(c-\frac{\omega}{\mu}s)}$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s\, e^{-i\pi\nu}(ks)^{\nu}\, e^{-ksz}ds=e^{-i\pi\nu}(kz^{\nu})^{-1}\,{}_2F_1(a,b,;c;\omega,\mu;-e^{-kz})$$

Case II. $\frac{\pi}{2}<|\arg z|<\pi$, we have

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu}=\underset{-}{C}\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu}$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{{}_2F_1(a,b;c;\omega,\mu;e^{-k\zeta})}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{\left\{\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega}{\mu}s)}{\Gamma(c-\frac{\omega}{\mu}s)}(e^{-k\zeta})^s ds\right\}}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds\left\{\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{e^{-ks\zeta}}{(\zeta-z)^{\nu+1}}d\zeta\right\}=e^{-i\pi\nu}(kz^{\nu})^{-1}\,{}_2F_1(a,b,;c;\omega,\mu;-e^{-kz})$$

Case III. $|\arg z|=\frac{\pi}{2}$

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu}=\underset{+}{C}\left({}_2F_1(a,b,;c;\omega,\mu;e^{-kz})\right)_{\nu}$$

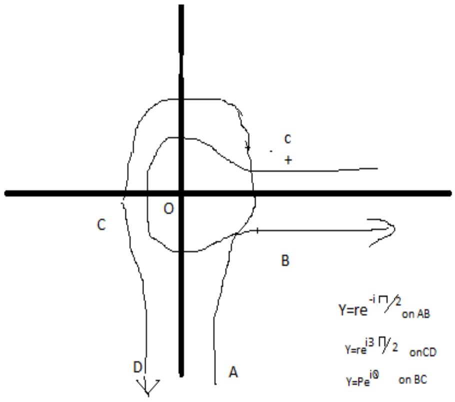
$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{{}_2F_1(a,b;c;\omega,\mu;e^{-k\zeta})}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{\left\{\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega}{\mu}s)}{\Gamma(c-\frac{\omega}{\mu}s)}(e^{-k\zeta})^s ds\right\}}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds\left\{\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{e^{-ks\zeta}}{(\zeta-z)^{\nu+1}}d\zeta\right\}$$

(put $\zeta-z=\eta, ks\eta=\xi, 0\le|\arg\eta|\le 2\pi$)

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds.(ks)^{\nu} e^{-ksz}\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty e^{-i\frac{\pi}{2}}}^{(0+)} \xi^{-(\nu+1)}e^{-\xi}d\xi,\ (\phi=\arg k=-\frac{\pi}{2}) \quad (2.1)$$



And

$$\int_{\infty e^{-i\frac{\pi}{2}}}^{(0+)} \xi^{-(\nu+1)}e^{-\xi}d\xi=\left(\int_{AB}+\int_{CD}+\int_{BC}\right)\xi^{-(\nu+1)}e^{-\xi}d\xi$$

$$=\int_{\infty}^{0}\left(re^{-i\frac{\pi}{2}}\right)^{-(\nu+1)} e^{-re^{-i\frac{\pi}{2}}}e^{-i\frac{\pi}{2}}dr+\int_{0}^{\infty}\left(re^{-i\frac{3\pi}{2}}\right)^{-(\nu+1)} e^{-re^{-i\frac{3\pi}{2}}}e^{-i\frac{3\pi}{2}}dr$$

$$+\lim_{\rho\to 0}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(\rho e^{i\theta}\right)^{-(\nu+1)} e^{-(\rho e^{i\theta})}\rho i e^{i\theta}d\theta$$

$$=-2ie^{-i\frac{\pi}{2}\nu}\sin\pi\nu\Gamma(-\nu)e^{-i\frac{\pi}{2}\nu}=-2\pi i e^{-i\pi\nu}\frac{\sin\pi\nu}{\pi}\Gamma(-\nu)=\frac{2\pi i e^{-i\pi\nu}}{\Gamma(\nu+1)}$$

From (2.1), we get

$=e^{-i\pi\nu}(kz^{\nu})^{-1}\,{}_2F_1(a,b,;c;\omega,\mu;-e^{-kz})$.

Theorem 2.

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{kz})\right)_{\nu}=\Gamma(\nu+1)(kz^{\nu})^{-1}\,{}_2F_1(a,b,;c;\omega,\mu;-e^{kz})\text{ for } k\neq 0(z,\nu\in C)$$

Proof: In case of $|\arg k|<\dfrac{\pi}{2}$

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{kz})\right)_{\nu}=\underset{-}{C}\left({}_2F_1(a,b,;c;\omega,\mu;e^{kz})\right)_{\nu}$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{{}_2F_1(a,b;c;\omega,\mu;e^{k\zeta})}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds\left\{\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{e^{ks\zeta}}{(\zeta-z)^{\nu+1}}d\zeta\right\}$$

(put $\zeta-z=\eta, ks\eta=\xi, 0\le|\arg\eta|\le 2\pi$)

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds.(ks)^{\nu} e^{ksz}\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty e^{-i\frac{\pi}{2}}}^{(0+)} \xi^{-(\nu+1)}e^{-\xi}d\xi,\ (\phi=\arg k)$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds.(ks)^{\nu} e^{ksz}\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty}^{(0+)}\xi^{-(\nu+1)}e^{-\xi}d\xi,\ (|\phi|<\frac{\pi}{2})$$

For $|\arg k|<\frac{\pi}{2}$, $\int_{-\infty}^{(0+)}\xi^{-(\nu+1)}e^{-\xi}d\xi=\frac{2\pi i}{\Gamma(\nu+1)}$

We arrive at the required result.

In case of $\frac{\pi}{2}\leq|\arg k|\leq\pi$, we have

$$\left({}_2F_1(a,b,;c;\omega,\mu;e^{kz})\right)_{\nu}=\underset{+}{C}\left({}_2F_1(a,b,;c;\omega,\mu;e^{kz})\right)_{\nu}$$

By using similar lines we can prove the result easily.

Corollary:

$$\left({}_2F_1(a,b,;c;\omega,\mu;k^z)\right)_{\nu}=\Gamma(\nu+1)(\log kz^{\nu})^{-1}{}_2F_1(a,b,;c;\omega,\mu;-k^z)\ \text{for}\ k\neq 0(z,\nu\in C)\ ()$$

Proof: We can write as

$$\left({}_2F_1(a,b,;c;\omega,\mu;k^z)\right)_{\nu}=\left({}_2F_1(a,b,;c;\omega,\mu;e^{z\log k})\right)_{\nu}$$

Some Examples:

(i) $${}_2F_1(a,b;c;\omega,\mu;e^{-5z})_{\frac{1}{2}}=e^{-i\frac{\pi}{2}}(5z^{\frac{1}{2}})^{-1}{}_2F_1(a,b;c;\omega,\mu;-e^{-5z})$$

$$=-\frac{i}{5\sqrt{z}}{}_2F_1(a,b;c;\omega,\mu;-e^{-5z})$$

(ii) $${}_2F_1(a,b;c;\omega,\mu;e^{-5z})_{-\frac{1}{2}}=-\frac{i\sqrt{z}}{5}{}_2F_1(a,b;c;\omega,\mu;-e^{-5z})$$

(iii) $${}_2F_1(a,b;c;\omega,\mu;e^{3z})_{\frac{1}{2}}=\frac{\sqrt{\pi}}{6\sqrt{z}}{}_2F_1(a,b;c;\omega,\mu;-e^{3z})$$

(iv) $${}_2F_1(a,b;c;\omega,\mu;e^{-3z})_{-\frac{1}{2}}=\frac{\sqrt{\pi z}}{3}{}_2F_1(a,b;c;\omega,\mu;-e^{-3z})$$

(v) $${}_2F_1(a,b;c;\omega,\mu;k^z)_{\frac{1}{2}}=\frac{\sqrt{\pi}}{2\log 3\sqrt{z}}{}_2F_1(a,b;c;\omega,\mu;-k^z)$$

$$==\frac{\sqrt{\pi}}{\log 9z}{}_2F_1(a,b;c;\omega,\mu;-k^z)$$

(vi) $${}_2F_1(a,b;c;\omega,\mu;k^z)_{-\frac{1}{2}}=\frac{\sqrt{\pi}}{2\log(3/\sqrt{z})}{}_2F_1(a,b;c;\omega,\mu;-k^z)$$

$$==\frac{\sqrt{\pi}}{\log(\frac{9}{z})}{}_2F_1(a,b;c;\omega,\mu;-k^z)$$

Theorem 3.

$${}_2F_1(a,b;c;\omega,\mu;z^k)_{\nu}=e^{-i\pi\nu}z^{-\nu}\frac{\Gamma(\nu-ks)}{\Gamma(-ks)}{}_2F_1(a,b;c;\omega,\mu;-z^k)$$

Case I: If $\left|\frac{\Gamma(\nu-ks)}{\Gamma(-ks)}\right|<\infty$, we have then

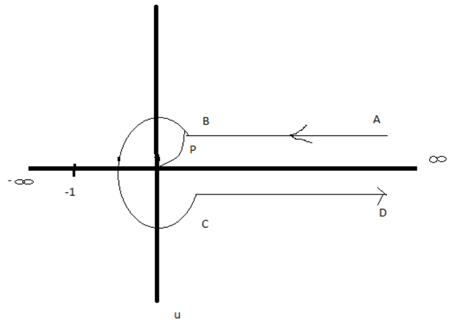
$${}_2F_1\left(a,b;c;\omega,\mu;z^k\right)_{\nu}=\underset{+}{C}\left({}_2F_1\left(a,b;c;\omega,\mu;z^k\right)\right)_{\nu}$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{C_+}\frac{{}_2F_1(a,b;c;\omega,\mu;\zeta^k)}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty e^{i\phi}}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} z^{ks-\nu}du,\ (\phi=\arg z)$$

By putting ($\zeta - z = \eta, \eta = zu$)

$$=\frac{1}{2\pi i}\int_L \theta(s)(-1)^s ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} z^{ks-\nu}du,(\phi<\frac{\pi}{2}) \qquad (2.2)$$



And

$$\int_{\infty}^{(0+)} u^{-(\nu+1)}(1+u)^{ks}du$$

$$=\lim_{\rho\to 0}\left(\int_{AB}+\int_{BC}+\int_{CD}\right)u^{-(\nu+1)}(1+u)^{ks}du \qquad (2.3)$$

$$(u=re^{i\theta} on\ AB, u=re^{i2\pi} on\, CD, u=\rho e^{i\theta}\ on\, BC)$$

$$=-\int_0^\infty r^{-(\nu+1)}(1+r)^{ks}dr + e^{-i2\pi\nu}\int_0^\infty r^{-(\nu+1)}(1+r)^{ks}dr+\lim_{\rho\to 0}\rho^{-\nu}\int_0^{2\pi}e^{-i\theta\nu}d\theta$$

$$=(e^{-i2\pi\nu}-1)\int_0^\infty\int_0^\infty r^{-(\nu+1)}(1+r)^{ks}dr,\ (\mathrm{Re}(\nu)<0) \qquad (2.4)$$

$$e^{-i2\pi\nu}-1=-i2e^{-i\pi\nu}\sin\pi\nu=e^{-i\pi\nu}\frac{2\pi i}{\Gamma(\nu+1)\Gamma(-\nu)} \qquad (2.5)$$

And $\int_0^\infty r^{-(\nu+1)}(1+r)^{ks}dr=\frac{\Gamma(-\nu)\Gamma(\nu-ks)}{\Gamma(-ks)}(\mathrm{Re}(ks)<\mathrm{Re}(\nu)<0)$ (2.6)

Applying (2.3), (2.6) into (2.4), we have then

$$\int_{\infty}^{(0+)} u^{-(\nu+1)}(1+u)^{ks}du=e^{-i\pi\nu}\frac{\Gamma(\nu-ks)}{\Gamma(-ks)}\frac{2\pi i}{\Gamma(\nu+1)} \qquad (2.7)$$

Substituting (2.7) into (2.2), we have then

$$=e^{-i\pi\nu}z^{-\nu}\frac{\Gamma(\nu-ks)}{\Gamma(-ks)}\,{}_2F_1(a,b;c;\omega,\mu;-z^k)$$

For $\mathrm{Re}(ks)<\mathrm{Re}(\nu)<0, |\arg z|<\frac{\pi}{2}, \left|\frac{\Gamma(\nu-ks)}{\Gamma(-ks)}\right|<\infty$.

Case II: For $\mathrm{Re}(ks) < \mathrm{Re}(\nu) < 0, \frac{\pi}{2} \leq |\arg z| \leq \pi, \left|\frac{\Gamma(\nu - ks)}{\Gamma(-ks)}\right| < \infty$

In the same way, we have

$$ {}_2F_1\left(a,b;c;\omega,\mu;z^k\right)_\nu = \underset{-}{C}\left({}_2F_1\left(a,b;c;\omega,\mu;z^k\right)\right)_\nu $$

$$ = \frac{1}{2\pi i}\int_L \theta(s)(-1)^s\, ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty e^{i\phi}}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} z^{ks-\nu} du,\ (\phi = \arg z) $$

$$ = \frac{1}{2\pi i}\int_L \theta(s)(-1)^s\, ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} z^{ks-\nu} du, (\frac{\pi}{2} < \phi < \pi) $$

$$ = e^{-i\pi\nu} z^{-\nu} \frac{\Gamma(\nu - ks)}{\Gamma(-ks)}\, {}_2F_1(a,b;c;\omega,\mu;-z^k) $$

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